NONLINEAR OSCILLATIONS IN A SELF-OSCILLATORY SYSTEM WITH TWO DEGREES OF FREEDOM

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INTRODUCTION

This paper deals with self-excited oscillations and forced oscillations in a self-oscillatory system with two degrees of freedom. A considerable number of papers have been published concerning the nonlinear systems with one degree of dreedom [1, 6, 10, 18, 20, 35].* However, few investigations have been reported on the nonlinear systems with two or more than two degrees of freedom. Among them, the systems having nonlinear restoring forces have been studied in some degree [2, 17, 29, 30, 33, 34, 36]; however, the behavior of self-oscillatory systems under the influence of an external force have been investigated very little [5, 21, 24].

The text consists of six chapters. In the first chapter the fundamental equations are derived from an electrical circuit with two degrees of freedom. The self-excited oscillations in this system are treated in the following two chapters. The last three chapters are concerned with forced oscillations in the self-oscillatory system with two degrees of freedom.

In Chapter 1 fundamental equations are derived for a negative-resistance oscillator having two resonant circuits. These circuits are inductively coupled and one of them is connected to a negative-resistance element. The driving source is applied to one of the resonant circuits. The system is described by two second-order differential equations, one of which contains a small nonlinear damping term. When the coupling between the two resonant circuits is not small, the system has two distinct natural frequencies. In this case fundamental equations are transformed to the standard form by means of a linear

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* Numbers in the brackets indicate references on pages 169 to 172.

transformation of the variables.

The two chapters that follow are devoted to the study of self-excited oscillations. The self-excited oscillations in a negative-resonance oscillator having two resonant circuits were treated first by van der Pol [25, 27]. There are a number of subsequent papers on the behavior of such systems [5, 7, 8, 11, 14, 19, 20, 22, 28]. It has usually been assumed that the coupling between the two resonant circuits is large and that there exist two distinct natural frequencies. This paper is intended to discuss the oscillations of the system not only for large coupling between the two resonant circuits but also for small coupling between them. In the latter case the entrainment of the two frequencies, that is, the internal resonance between them, occurs if they are close each other [3, 16]. More generally, the internal resonance of the system occurs when the two frequencies have a certain relationship [9, 13, 32].

Chapter 2 is concerned with the self-excited oscillations in the case where two distinct natural frequencies exist and the ratio of them is not in the neighborhood of an integer or a fraction. In this case no internal resonance occurs. The solution is assumed in the form of a sinusoidal oscillation in which the amplitude and the phase angle vary slowly with time. By using the averaging method the equations of the standard form are transformed into an autonomous system [4, 5]. The amplitude characteristics of the oscillations are obtained from the states of equilibrium of this autonomous system. The stability of these oscillations is discussed by solving variational equations which characterize small deviations from the states of equilibrium. The variational equations are linear differential equations with constant coefficients. Therefore, the stability conditions are obtained by making use of the

Routh-Hurwitz criterion [15, 31]. The results thus obtained secure the stability of two types of periodic oscillations, each having one of the natural frequencies. On the other hand, the combination oscillation having both of them, which is generally almost periodic, is unstable. The phase-plane analysis is applied to the study of the transient states of the oscillations [10]. The coordinates of the phase plane are the time-varying amplitudes of the two types of periodic oscillations, each having one of the natural frequencies. The steady states of the oscillations are represented by singular points in the phase plane. There exist the integral curves that divide the phase plane into two domains of attraction, each containing a stable singular point. Hence once the initial conditions of the system are given, the resulting response is found.

In Chapter 3 two representative cases of the internal resonance in the self-excited oscillations are studied. The first is the case in which the ratio of the two natural frequencies is in the neighborhood of 1/3. The second case deals with the internal resonance which occurs when the two natural frequencies are close each other.

In the former case, the entrainment occurs between the higher frequency of the two natural frequencies and the frequency which is three times multiple of the lower one. The response is an entrained periodic oscillation which has the third-harmonic component predominantly. The harmonic and the higher-harmonic frequencies of the entrained oscillation are not equal to the natural frequencies but are in the neighborhood of them. To investigate this phenomenon, the unknown frequencies of the entrained oscillation are introduced. By using the averaging method, the steady-state solutions are sought and their stability is discussed.

The latter case occurs, as mentioned earlier, when the coupling between the two resonant circuits of the oscillator is small and their natural frequencies are close each other. Under this condition, the entrainment of the two frequencies occurs. The same method of analysis as mentioned above is used. The frequency, amplitude, and phase characteristics of the entrained oscillation are calculated under varying frequencies of the resonant circuits. It is found that only one type of periodic oscillation exists when the coupling is very small.

The last three chapters are devoted to a study of the oscillations which occur when a periodic excitation is applied to the self-oscillatory system. The phenomenon of frequency entrainment occurs when a periodic force is applied to a self-oscillatory system with one degree of freedom. A typical and important case is the system governed by van der Pol's equation with an additional term representing a periodic excitation. The behavior of such systems is treated in a considerable number of papers [6, 10, 26, 27, 35]. In this paper the behavior of a self-oscillatory system with two degrees of freedom under the influence of an external periodic force is discussed.

Chapter 4 treats forced oscillations in a self-oscillatory system in which no internal resonance occurs [5, 12, 21, 23, 24]. As mentioned in Chapter 2 this system produces two types of self-excited oscillations in the absence of the external force. Under the impression of a periodic force, one of the frequencies of the self-excited oscillations falls in synchronism with the driving frequency within a certain band of frequencies, that is, the external resonance occurs. This phenomenon of frequency entrainment also occurs when the ratio of one of the natural frequencies and the driving frequency is in the neighborhood of an integer (different from unity) or a fraction. Under this condition, one of natural frequencies of the system is entrained by a frequency which is an integral multiple or submultiple of the driving frequency. In these cases, if the component having the other natural frequency does not build up, the resulting oscillation becomes periodic. On the other hand, if the component having the other natural frequency builds up, the oscillation is almost periodic. Moreover, when the driving frequency is in the neighborhood of a frequency which is a linear combination of the two natural frequencies, a kind of resonance occurs, such that the linear combination of the natural frequencies is entrained by the driving frequency [12, 24]. The entrained oscillation contains three frequency components and is generally almost periodic. When no external resonance occurs, the system produces two types of oscillations represented by a sum of the components having the driving frequency and one of the natural frequencies. These are generally almost periodic.

The averaging method is applied to the solution of the fundamental equations of the standard form. According to the types of external resonance, autonomous systems of the equations are derived. The amplitude and phase characteristics of the entrained oscillations are obtained from these systems. The stability of the oscillations is investigated by making use of the Routh-Hurwitz criterion. The regions in which different types of entrained oscillations and almost periodic oscillations occur are sought under varying the amplitude and frequency of the external force.

Chapter 5 deals with the forced oscillations in a self-oscillatory system in which the ratio between the two natural frequencies is in the neighborhood of an integer (different from unity) or a fraction [13, 36]. In this case the internal resonance occurs. If one of the two natural frequencies is in the neighborhood of an integral multiple or submultiple of the driving frequency, so is the other natural frequency. Therefore, both of the two natural frequencies are entrained by the frequencies which are integral multiples or submultiples of the driving frequency. The entrained oscillations which consist of three harmonic components are periodic. The same method of analysis as in the preceding chapters is used for the investigation of the steady-state solutions and their stability.

The last Chapter 6 is concerned with the forced oscillations in a self-oscillatory system with the internal resonance which occurs when the two natural frequencies are close each other. The averaging method is applied to the study of the entrained oscillations. The phenomenon of frequency entrainment occurs at harmonic, higher-harmonic, or subharmonic frequency of the external force. As mentioned in Chapter 3, when the coupling between the two resonant circuits is very small, the system under consideration has only one kind of self-excited oscillation in the absence of the external force. Therefore, the amplitude characteristics of the entrained oscillations under this condition are similar to those of van der Pol's equation with a forcing term. As the coupling becomes a little larger, two types of self-excited oscillations occur, each of them being entrained by the driving frequency. A special attention is directed to the transition of the behavior of the oscillatory system under varying the coupling between the two resonant circuits.

The text is supplemented by three appendixes. In Appendix I is given an expanded form of the fundamental equations from which the autonomous systems are derived by using the averaging method. In Appendix II is given the stability condition derived from the Routh-Hurwitz criterion, and is shown that the characteristic curve has a vertical tangent at the stability limit [10]. Appendix III derives the conditions for self-excitation of the self-oscillatory system.

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CONTENTS

CHAPTER 1

DERIVATION OF THE FUNDAMENTAL EQUATIONS OF A SELF-OSCILLATORY SYSTEM WITH TWO DEGREES OF FREEDOM

1.1 Introduction

In this chapter we derive the differential equations which describe the self-excited oscillations and forced oscillations in a negative-resistance oscillator having two resonant circuits. These circuits are inductively coupled, and one of them is connected to a negative-resistance element. When the coupling between the two resonant circuits is not so small, this system has two distinct natural frequencies. In this case, it is convenient to transform the differential equations into the standard form. The analysis of these equations is performed in the following chapters.

1.2 Derivation of the Fundamental Equations

The schematic diagram illustrated in Fig. 1.1 shows a vacuum-tube oscillator which contains two resonant circuits $L_1R_1C_1$ and $L_2R_2C_2$. We assume that the inductive coupling exists between L_1 and L_2, between L_1 and L_3, but no coupling between L_2 and L_3, and that the external force is applied in series with L_1. Using Kirchhoff's laws and neglecting the grid current of the vacuum tube, we write the equations of the circuits as

$$
\begin{aligned}
&L_1 \frac{di_1}{dt} + R_1 i_1 + e_1 = M \frac{di_a}{dt} + N \frac{di_2}{dt} + E \cos \omega t \\
&L_2 \frac{di_2}{dt} + R_2 i_2 + e_2 = N \frac{di_1}{dt} \qquad\qquad (1.1) \\
&e_1 = \frac{1}{C_1} \int i_1 dt, \qquad e_2 = \frac{1}{C_2} \int i_2 dt
\end{aligned}
$$

Eliminating i_1 and i_2 in Eqs. (1.1), we obtain

$$
\begin{aligned}
L_1C_1 \frac{d^2e_1}{dt^2} + C_1R_1 \frac{de_1}{dt} + e_1 &= M \frac{di_a}{dt} + NC_2 \frac{d^2e_2}{dt^2} + E \cos \omega t \\
L_2C_2 \frac{d^2e_2}{dt^2} + C_2R_2 \frac{de_2}{dt} + e_2 &= NC_1 \frac{d^2e_1}{dt^2}
\end{aligned}
\tag{1.2}
$$

Setting

$$
\begin{aligned}
n_1^2 &= \frac{1}{L_1C_1}, \qquad & n_2^2 &= \frac{1}{L_2C_2} \\
\chi_1 &= \frac{NC_2}{L_1C_1}, \qquad & \chi_2 &= \frac{NC_1}{L_2C_2}
\end{aligned}
\tag{1.3}
$$

we obtain

$$
\begin{aligned}
\frac{d^2e_1}{dt^2} - \chi_1 \frac{d^2e_2}{dt^2} + n_1^2e_1 &= n_1^2\left(- C_1R_1 \frac{de_1}{dt} + M \frac{di_a}{dt} + E \cos \omega t\right) \\
\frac{d^2e_2}{dt^2} - \chi_2 \frac{d^2e_1}{dt^2} + n_2^2e_2 &= - n_2^2C_2R_2 \frac{de_2}{dt}
\end{aligned}
\tag{1.4}
$$

If we neglect the plate reaction and assume that the plate current-grid voltage characteristic is a cubic, then

$$
i_a = S_1e_1 - \frac{1}{3} S_3e_1^3 \tag{1.5}
$$

where S_1 and S_3 are certain constants characterizing the vacuum tube. The nonlinearity is introduced in Eqs. (1.4) owing to this nonlinear characteristic. Introducing the dimentionless variables u and v defined by

$$
u = e_1\sqrt{\frac{MS_3}{MS_1 - C_1R_1}}, \qquad v = e_2\sqrt{\frac{MS_3}{MS_1 - C_1R_1}} \tag{1.6}
$$

and using Eq. (1.5), we rewrite Eqs. (1.4) in the form

$$\begin{aligned} \ddot{u} - \chi_1\ddot{v} + n_1^2 u &= \mu n_1(1 - u^2)\dot{u} + n_1^2 B \cos \omega t \\ \ddot{v} - \chi_2\ddot{u} + n_2^2 v &= -\mu \frac{n_2^2}{n_1} \delta\dot{v} \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \mu &= (MS_1 - C_1R_1)n_1 \\ \delta &= \frac{C_2R_2}{MS_1 - C_1R_1} \\ B &= E\sqrt{\frac{MS_3}{MS_1 - C_1R_1}} \end{aligned} \tag{1.8}$$

The dots over u and v refer to differentiation with respect to t. Equations (1.7) are the fundamental equations that describe the behavior of the system of Fig. 1.1.

Setting B = 0 in Eqs. (1.7), we obtain the following fundamental equations which describe the self-excited oscillations.

$$\begin{aligned} \ddot{u} - \chi_1\ddot{v} + n_1^2 u &= \mu n_1(1 - u^2)\dot{u} \\ \ddot{v} - \chi_2\ddot{u} + n_2^2 v &= -\mu \frac{n_2^2}{n_1} \delta\dot{v} \end{aligned} \tag{1.9}$$

When N = 0, i.e., $\chi_1 = \chi_2 = 0$ in Eqs. (1.9), the first equation of (1.9) is reduced to van der Pol's equation, and the second equation describes a damped oscillation in the $L_2R_2C_2$-circuit. Since the system is self-oscillatory, μ must be positive. We assume hereafter that μ is a small positive quantity. Moreover we assume that the resistance R_2 is as small as μ, therefore the parameter δ is not so large.

1.3 Transformation of the Fundamental Equations to the Standard Form

(a) The Standard Form of Equations Describing the Self-Excited Oscillations

Further modification of the differential equations (1.7) and (1.9) is useful for the subsequent analysis. First, we consider the self-oscillatory system (1.9). Neglecting the small term of order μ in Eqs. (1.9), we obtain the so called GENERATING system:

$$\begin{aligned} \ddot{u} - \chi_1 \ddot{v} + n_1^2 u &= 0 \\ \ddot{v} - \chi_2 \ddot{u} + n_2^2 v &= 0 \end{aligned} \tag{1.10}$$

The general solution of this system, i.e., the GENERATING solution takes the form

$$\begin{aligned} u(t) &= r_1 \cos(\omega_1 t + \theta_1) + r_2 \cos(\omega_2 t + \theta_2) \\ v(t) &= k_1 r_1 \cos(\omega_1 t + \theta_1) + k_2 r_2 \cos(\omega_2 t + \theta_2) \end{aligned} \tag{1.11}$$

where r_i, θ_i $(i = 1, 2)$ are integration constants, ω_i are the positive roots of the following equation*

$$(1 - \chi_1 \chi_2)\omega^4 - (n_1^2 + n_2^2)\omega^2 + n_1^2 n_2^2 = 0 \tag{1.12}$$

and k_i are given by

$$k_i = \frac{\omega_i^2 - n_1^2}{\chi_1 \omega_i^2} = \frac{\chi_2 \omega_i^2}{\omega_i^2 - n_2^2} \tag{1.13}$$

ω_i are called as the natural frequencies of the system.

---

* Without loss of generality we may assume that $\omega_1 < \omega_2$. Then we obtain the relations: $\omega_1 < n_1 < \omega_2$, $\omega_1 < n_2 < \omega_2$, $k_1 < 0$ and $k_2 > 0$.

From Eqs. (1.11) it follows that the generating system has a general solution with two frequency components ω_1 and ω_2. Hence, for a small value of μ, we may assume that the solution of Eqs. (1.9) take a form similar to Eqs. (1.11). Therefore we write

$$\begin{aligned} u &= x + y \\ v &= k_1 x + k_2 y \end{aligned} \tag{1.14}$$

Substituting Eqs. (1.14) into Eqs. (1.9) leads to the standard form of equations

$$\begin{aligned} \ddot{x} + \omega_1^2 x &= \mu F_1(x,\ y,\ \dot{x},\ \dot{y}) \\ \ddot{y} + \omega_2^2 y &= \mu G_1(x,\ y,\ \dot{x},\ \dot{y}) \end{aligned} \tag{1.15}$$

where

$$\begin{aligned} F_1(x,\ y,\ \dot{x},\ \dot{y}) &= \frac{\omega_1^2}{n_1} \frac{1}{k_2 - k_1} \left\{k_2[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y})\right\} \\ G_1(x,\ y,\ \dot{x},\ \dot{y}) &= \frac{\omega_2^2}{n_1} \frac{1}{k_1 - k_2} \left\{k_1[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y})\right\} \end{aligned} \tag{1.16}$$

Equations (1.15) are the standard form of the fundamental equations (1.9). It is to be noted that, if ω_1 and ω_2 are sufficiently close each other, one may expect the internal resonance, i.e., the entrainment between these frequencies owing to the nonlinearity of the system. In this case the present analysis ceases to be meaningful. Since the discriminant of Eq. (1.12) is given by $(n_1^2 - n_2^2)^2 + 4\chi_1\chi_2 n_1^2 n_2^2$, this type of internal resonance occurs when $n_1 \cong n_2$ and $N \cong 0$ [see Eqs. (1.3)]. This case will be discussed in Sec. 3.3.

(b) The Standard Form of Equations Describing the Forced Oscillations

We consider the fundamental equations (1.7) describing the forced oscillations. The generating system of Eqs. (1.7) is

$$\ddot{u} - \chi_1\ddot{v} + n_1^2 u = n_1^2 B \cos \omega t$$
$$\ddot{v} - \chi_2\ddot{u} + n_2^2 v = 0 \tag{1.17}$$

When the driving frequency ω is not equal to the natural frequency ω_1 nor ω_2, the general solution is given by

$$u(t) = r_1 \cos(\omega_1 t + \theta_1) + r_2 \cos(\omega_2 t + \theta_2) + A_1 \cos \omega t$$
$$v(t) = k_1 r_1 \cos(\omega_1 t + \theta_1) + k_2 r_2 \cos(\omega_2 t + \theta_2) + A_2 \cos \omega t \tag{1.18}$$

where r_i, θ_i $(i = 1, 2)$ are integration constants, ω_i and k_i are determined by Eqs. (1.12) and (1.13), and A_i are given by

$$A_1 = \frac{n_1^2(n_2^2 - \omega^2)}{(1 - \chi_1\chi_2)(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} B$$
$$A_2 = \frac{-\chi_2 n_1^2 \omega^2}{(1 - \chi_1\chi_2)(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \cdot B \tag{1.19}$$

Hence, proceeding in the same manner as in Eqs. (1.14) we assume the solution of Eqs. (1.7) in the form*

$$u = x + y + A_1 \cos \omega t$$
$$v = k_1 x + k_2 y + A_2 \cos \omega t \tag{1.20}$$

Substituting Eqs. (1.20) into Eqs. (1.7) gives the standard form of equations.

---

* The term containing $\sin \omega t$ is omitted in the solution for the first approximation (1.20).

$$\ddot{x} + \omega_1^2 x = \mu F_2(x,\ y,\ \dot{x},\ \dot{y},\ t)$$
$$\ddot{y} + \omega_2^2 y = \mu G_2(x,\ y,\ \dot{x},\ \dot{y},\ t) \tag{1.21}$$

where

$$F_2(x,\ y,\ \dot{x},\ \dot{y},\ t) = \frac{\omega_1^2}{n_1}\frac{1}{k_2 - k_1}\Big\{k_2[1 - (x + y + A_1 \cos \omega t)^2](\dot{x} + \dot{y} - \omega A_1 \sin \omega t) + \delta(k_1\dot{x} + k_2\dot{y} - \omega A_2 \sin \omega t)\Big\}$$

$$G_2(x,\ y,\ \dot{x},\ \dot{y},\ t) = \frac{\omega_2^2}{n_1}\frac{1}{k_1 - k_2}\Big\{k_1[1 - (x + y + A_1 \cos \omega t)^2](\dot{x} + \dot{y} - \omega A_1 \sin \omega t) + \delta(k_1\dot{x} + k_2\dot{y} - \omega A_2 \sin \omega t)\Big\} \tag{1.22}$$

When the driving frequency ω is in the neighborhood of either ω_1 or ω_2, one may expect the external resonance. Namely, the natural frequency is entrained by the driving frequency ω. In fact, when the difference $\omega - \omega_i$ $(i = 1,\ 2)$ is equal to zero, the denominators of A_1 and A_2 in Eqs. (1.19) become zero. In this case the assumption of the solution in the form (1.20) is not valid. Therefore, when ω and one of the natural frequencies are sufficiently close, the amplitude B of the driving force is of the order μ, i.e.,

$$B = \mu B_1 \tag{1.23}$$

Then, the fundamental equations (1.7) are rewritten as

$$\ddot{u} - \chi_1\ddot{v} + n_1^2 u = \mu[n_1(1 - u^2)\dot{u} + n_1^2 B_1 \cos \omega t]$$
$$\ddot{v} - \chi_2\ddot{u} + n_2^2 v = -\mu\frac{n_2^2}{n_1}\delta\dot{v} \tag{1.24}$$

The generating solution of Eqs. (1.24) takes the same form as Eqs. (1.11). Introducing x and y defined by Eqs. (1.14), we obtain the standard form of equations that describe the external resonance $\omega \cong \omega_i$.

$$\ddot{x} + \omega_1^2 x = \mu F_3(x, y, \dot{x}, \dot{y}, t)$$
$$\ddot{y} + \omega_2^2 y = \mu G_3(x, y, \dot{x}, \dot{y}, t) \tag{1.25}$$

where

$$F_3(x, y, \dot{x}, \dot{y}, t) = \frac{\omega_1^2}{n_1}\frac{1}{k_2 - k_1}\left\{k_2[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y}) + k_2 n_1 B_1 \cos \omega t\right\}$$
$$G_3(x, y, \dot{x}, \dot{y}, t) = \frac{\omega_2^2}{n_1}\frac{1}{k_1 - k_2}\left\{k_1[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y}) + k_1 n_1 B_1 \cos \omega t\right\} \tag{1.26}$$

Obviously, putting $B = B_1 = 0$, i.e., $A_1 = A_2 = 0$ in Eqs. (1.21) and (1.25) yields the standard form of self-oscillatory system (1.15). As mentioned before in the paragraph of the self-oscillatory system, if two natural frequencies ω_1 and ω_2 are sufficiently close each other, the present transformation of the fundamental equations to the standard form ceases to be meaningful.

In Chaps. 2 and 3, we discuss the self-excited oscillations based on Eqs. (1.9) and (1.15). In Chaps. 4, 5, and 6, we discuss the forced oscillations in a self-oscillatory system based on Eqs. (1.7), (1.21), and (1.25).

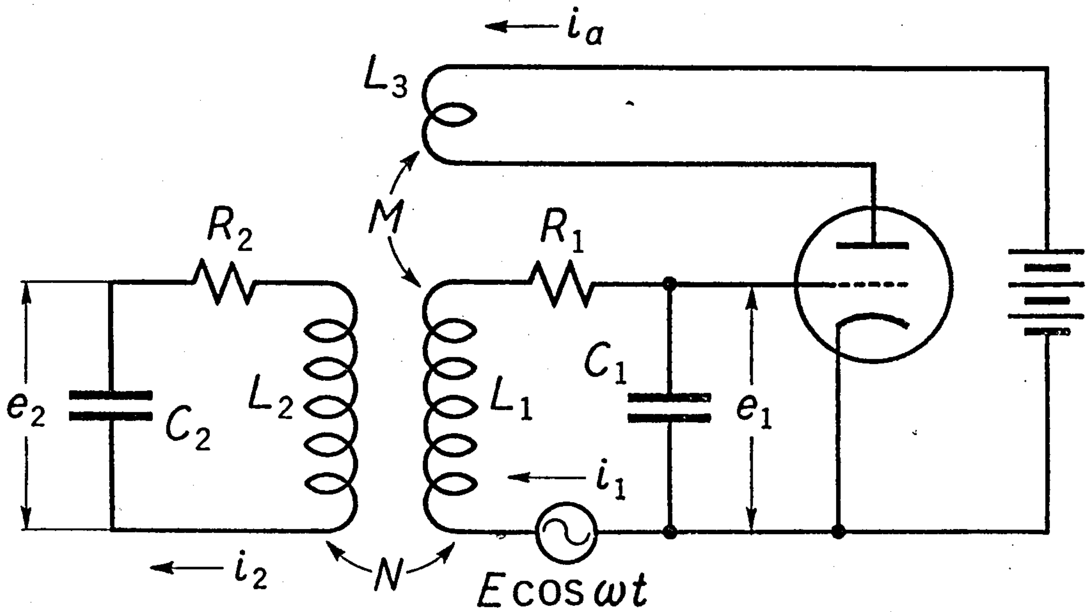


Fig. 1.1. Vacuum-tube oscillator with two resonant circuits and an external force.

CHAPTER 2

SELF-EXCITED OSCILLATIONS IN A SYSTEM WITH TWO DEGREES OF FREEDOM

2.1 Introduction

This chapter deals with self-excited oscillations in a negative-resistance oscillator having two resonant circuits. The coupling between the two resonant circuits is assumed to be large. Under this condition, there exist two distinct natural frequencies. In this chapter, we also assume that the ratio of the two natural frequencies is not in the neighborhood of an interger or a fraction, that is, no internal resonance occurs [5, 8, 11, 19, 20, 22, 25, 27].

By using the averaging method, an autonomous system is obtained from the standard form of equations which are derived in Chapter 1. The steady state of the oscillations and their stability are discussed. Particular attention is directed to the investigations of the transient state of the oscillations by applying the phase-plane analysis [5, 28].

2.2 Derivation of an Autonomous System by Using the Averaging Method

First, the standard form of equations (1.15) is transformed to a form suitable for the application of the averaging method. When $\mu = 0$, the solution of Eqs. (1.10) is given by Eqs. (1.11). Bearing this in mind, we write for x and y in Eqs. (1.15) as

$$\begin{aligned} x(t) &= r_1(t) \cos [\omega_1 t + \theta_1(t)] \\ y(t) &= r_2(t) \cos [\omega_2 t + \theta_2(t)] \end{aligned} \tag{2.1}$$

$$\begin{aligned} \dot{x}(t) &= -\omega_1 r_1(t) \sin [\omega_1 t + \theta_1(t)] \\ \dot{y}(t) &= -\omega_2 r_2(t) \sin [\omega_2 t + \theta_2(t)] \end{aligned} \tag{2.2}$$

We assume that, for small values of μ, both the amplitudes $r_1(t)$, $r_2(t)$ and the phase angles $\theta_1(t)$, $\theta_2(t)$ are slowly varying functions of t. Substituting Eqs. (2.1) and (2.2) into Eqs. (1.15) yields

$$\begin{aligned} \dot{r}_1 \sin(\omega_1 t + \theta_1) + r_1\dot{\theta}_1 \cos(\omega_1 t + \theta_1) &= -\frac{\mu}{\omega_1} f_1(r_1, r_2, \theta_1, \theta_2, t) \\ \dot{r}_2 \sin(\omega_2 t + \theta_2) + r_2\dot{\theta}_2 \cos(\omega_2 t + \theta_2) &= -\frac{\mu}{\omega_2} g_1(r_1, r_2, \theta_1, \theta_2, t) \end{aligned} \tag{2.3}$$

where $f_1(r_1, r_2, \theta_1, \theta_2, t)$ and $g_1(r_1, r_2, \theta_1, \theta_2, t)$ in the right-hand sides of Eqs. (2.3) are obtained by the substitution of Eqs. (2.1) and (2.2) into Eqs. (1.16).* We obtain from Eqs. (2.1) and (2.2) that

$$\begin{aligned} \dot{r}_1 \cos(\omega_1 t + \theta_1) - r_1\dot{\theta}_1 \sin(\omega_1 t + \theta_1) &= 0 \\ \dot{r}_2 \cos(\omega_2 t + \theta_2) - r_2\dot{\theta}_2 \sin(\omega_2 t + \theta_2) &= 0 \end{aligned} \tag{2.4}$$

Solving Eqs. (2.3) and (2.4) for the derivatives $\dot{r}_1$, $\dot{r}_2$, $\dot{\theta}_1$, and $\dot{\theta}_2$ gives

$$\begin{aligned} \dot{r}_1 &= -\frac{\mu}{\omega_1} f_1(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_1 t + \theta_1) \\ \dot{r}_2 &= -\frac{\mu}{\omega_2} g_1(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_2 t + \theta_2) \\ r_1\dot{\theta}_1 &= -\frac{\mu}{\omega_1} f_1(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_1 t + \theta_1) \\ r_2\dot{\theta}_2 &= -\frac{\mu}{\omega_2} g_1(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_2 t + \theta_2) \end{aligned} \tag{2.5}$$

Equations (2.5) show that $\dot{r}_i$ and $\dot{\theta}_i$ $(i = 1, 2)$ are both of the order of μ; therefore, as expected, r_i and θ_i are slowly varying functions of t. Hence upon application of the averaging method, Eqs. (2.5) can be transformed into

---

* Expansions of the functions f_1 and g_1 are shown in Appendix I.

an autonomous system, i.e.,

$$
\begin{aligned}
\dot{r}_1 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_1} f_1(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_1 t + \theta_1)dt \\
\dot{r}_2 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_2} g_1(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_2 t + \theta_2)dt \\
r_1\dot{\theta}_1 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_1} f_1(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_1 t + \theta_1)dt \\
r_2\dot{\theta}_2 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_2} g_1(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_2 t + \theta_2)dt
\end{aligned} \tag{2.6}
$$

The integration is to be performed with respect to the explicitly appearing t in the integrants. Performing the integration gives us

$$
\begin{aligned}
\dot{r}_1 &= \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - r_1^2 - 2r_2^2)r_1 \\
\dot{r}_2 &= \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2r_1^2 - r_2^2)r_2 \\
\dot{\theta}_1 &= 0 \\
\dot{\theta}_2 &= 0
\end{aligned} \tag{2.7}
$$

where

$$
\begin{aligned}
\rho_1 &= 4\left(1 + \frac{k_1}{k_2}\delta\right) = 4\left(1 - \frac{\omega_1^2 - n_1^2}{\omega_1^2 - n_2^2}\frac{n_2^2}{n_1^2}\delta\right) \\
\rho_2 &= 4\left(1 + \frac{k_2}{k_1}\delta\right) = 4\left(1 - \frac{\omega_2^2 - n_1^2}{\omega_2^2 - n_2^2}\frac{n_2^2}{n_1^2}\delta\right)
\end{aligned} \tag{2.8}
$$

If we write $r_i^2 = R_i$ $(i = 1, 2)$, the first two equations of (2.7) can be rewritten as

$$
\dot{R}_1 = \frac{\mu\omega_1^2}{4n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - R_1 - 2R_2)R_1 \equiv X(R_1, R_2)
$$

$$\dot{R}_2 = \frac{\mu\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2R_1 - R_2)R_2 \equiv Y(R_1, R_2) \tag{2.9}$$

Equations (2.7) and (2.9) play a significant role in the following investigation, since they serve as the fundamental equations in studying the transient state as well as the steady state.

If there exists a certain relationship between the two frequencies ω_1 and ω_2, additional terms may appear in the right-hand sides of Eqs. (2.7). The functions f_1 and g_1 in Eqs. (2.5) contain terms of several frequencies (see Appendix I), and some of them which apparently look different from ω_1 or ω_2 may coincide with these frequencies. For instance, if $3\omega_1 = \omega_2$, the frequency components $\omega_2 - 2\omega_1$ and $3\omega_1$ which are contained in f_1 and g_1 coincide with ω_1 and ω_2, respectively. Therefore these terms do not vanish upon integration in Eqs. (2.6). Similarly to the foregoing case where $\omega_1 \simeq \omega_2$, the internal resonance also occurs when $3\omega_1$ is in the neighborhood of ω_2. This type of internal resonance will be discussed in Sec. 3.2.

2.3 Self-Excited Oscillations in the Steady State

(a) Steady-State Solutions

Let us consider the steady-state solutions of Eqs. (2.7) in which r_i and θ_i (i = 1, 2) are constant. The last two equations of (2.7) show that the phase angles, θ_1 and θ_2 are constant. From Eqs. (2.9), we have

$$\begin{aligned} (\rho_1 - R_{10} - 2R_{20})R_{10} &= 0 \\ (\rho_2 - 2R_{10} - R_{20})R_{20} &= 0 \end{aligned} \tag{2.10}$$

where R_{10} and R_{20} denote the steady-state values of R_1 and R_2, respectively.

We see, from Eqs. (2.10), that there are four different states of equilibrium, i.e.,

$$
\begin{aligned}
&(1)\ R_{10} = 0, && R_{20} = 0 \\
&(2)\ R_{10} = \rho_1, && R_{20} = 0 \\
&(3)\ R_{10} = 0, && R_{20} = \rho_2 \\
&(4)\ R_{10} = \frac{1}{3}(2\rho_2 - \rho_1), && R_{20} = \frac{1}{3}(2\rho_1 - \rho_2)
\end{aligned}
\tag{2.11}
$$

No self-excited oscillation exists in (1); while, in (2) and (3), the solutions are periodic of frequency ω_1 and ω_2, respectively. Since ω_1 and ω_2 are generally incommensurable, the solution in (4) is almost periodic.

(b) Stability Investigation

The steady-state solutions given by Eqs. (2.11) are maintained only when they are stable. The stability of the solutions is tested by the behavior of small variations, ξ and η, from the steady-state values, R_{10} and R_{20}, respectively; i.e.,

$$R_1 = R_{10} + \xi, \qquad R_2 = R_{20} + \eta \tag{2.12}$$

If ξ and η approach zero with increase of time t, the steady-state solution is stable. Substituting Eqs. (2.12) into (2.9) gives*

$$
\begin{aligned}
\dot{\xi} &= a_{11}\xi + a_{12}\eta \\
\dot{\eta} &= a_{21}\xi + a_{22}\eta
\end{aligned}
\tag{2.13}
$$

where

$$a_{11} = \left(\frac{\partial X}{\partial R_1}\right)_0 = \frac{\mu\omega_1^2}{4n_1}\,\frac{k2}{k_2 - k_1}\,(\rho_1 - 2R_{10} - 2R_{20})$$

---

* Terms of degree higher than the first in ξ and η are neglected.

$$a_{12} = \left(\frac{\partial X}{\partial R_1}\right)_0 = -\frac{\mu\omega_1^2}{2n_1}\frac{k_2}{k_2 - k_1}R_{10}$$

$$a_{21} = \left(\frac{\partial Y}{\partial R_1}\right)_0 = -\frac{\mu\omega_2^2}{2n_1}\frac{k_1}{k_1 - k_2}R_{20} \tag{2.14}$$

$$a_{22} = \left(\frac{\partial Y}{\partial R_2}\right)_0 = \frac{\mu\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2R_{10} - 2R_{20})$$

The symbol $(\ )_0$ denotes the insertion of the steady-state values, R_{10} and R_{20}, after differentiation. The characteristic equation of the system (2.13) is given by

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 + p\lambda + q = 0 \tag{2.15}$$

where

$$p = -(a_{11} + a_{22}) = -\frac{\mu}{4n_1}\left[\frac{\omega_1^2 k_2}{k_2 - k_1}(\rho_1 - 2R_{10} - 2R_{20}) + \frac{\omega_2^2 k_1}{k_1 - k_2}(\rho_2 - 2R_{10} - 2R_{20})\right]$$

$$q = a_{11}a_{22} - a_{12}a_{21} = -\frac{\mu^2\omega_1^2\omega_2^2}{16n_1^2}\frac{k_1 k_2}{(k_1 - k_2)^2}\left[(\rho_1 - 2R_{10} - 2R_{20})(\rho_2 - 2R_{10} - 2R_{20}) - 4R_{10}R_{20}\right] \tag{2.16}$$

The variations ξ and η approach zero with the time t, provided that the real parts of λ's are negative. The stability conditions are given by the Routh-Hurwitz criterion (see Appendix II); i.e.,

$$p = -(a_{11} + a_{22}) > 0 \quad \text{and} \quad q = a_{11}a_{22} - a_{12}a_{21} > 0 \tag{2.17}$$

Substituting Eqs. (2.16) into (2.17), we obtain the stability conditions for the equilibrium states represented by Eqs. (2.11); i.e.,

(1) For $R_{10} = R_{20} = 0$, the conditions for stability are

$$\frac{\omega_1^2 k_2}{k_2 - k_1} \rho_1 + \frac{\omega_2^2 k_1}{k_1 - k_2} \rho_2 < 0 \qquad \text{and} \qquad k_1 k_2 \rho_1 \rho_2 < 0$$

Since $k_1 < 0$ and $k_2 > 0$, the above inequalities lead to

$$\rho_1 < 0 \qquad \text{and} \qquad \rho_2 < 0 \tag{2.18}$$

(2) For $R_{10} = \rho_1$, $R_{20} = 0$, we obtain

$$\rho_1 > \frac{1}{2} \rho_2 \qquad \text{and} \qquad \rho_1 > 0 \tag{2.19}$$

(3) For $R_{10} = 0$, $R_{20} = \rho_2$, we obtain

$$\rho_2 > \frac{1}{2} \rho_1 \qquad \text{and} \qquad \rho_2 > 0 \tag{2.20}$$

(4) For $R_{10} = \frac{1}{3} (2\rho_2 - \rho_1)$, $R_{20} = \frac{1}{3} (2\rho_1 - \rho_2)$, we obtain

$$R_{10} R_{20} < 0 \tag{2.21}$$

There exists no oscillation in case (1). Since both R_{10} and R_{20} are positive, condition (2.21) is never fulfilled. Hence, in the steady state, the oscillation is periodic with frequency ω_1 [case (2)] or frequency ω_2 [case (3)], and no combination oscillation of the two frequencies is realized [7,14].

(c) Numerical Examples

The coupling k between the two resonant circuits of Fig. 1.1 is given by [see Eqs. (1.3)]

$$k = \sqrt{N^2/L_1 L_2} = \sqrt{\chi_1 \chi_2} \tag{2.22}$$

By solving Eqs. (1.12) for a given value of k, we obtain ω/n_1 as a function of n_2/n_1. An example of such a frequency characteristic is shown in Fig. 2.1 where $k = 0.5$. If we furthermore assume δ or $(n_2/n_1)^2\delta$ in Eqs. (2.8), ρ_1 and ρ_2 are calculated.* Then, by using Eqs. (2.11), the amplitude characteristic of the self-excited oscillation is obtained. Calculated results are shown in Figs. 2.2 and 2.3.† The stability of the oscillation is tested by using conditions (2.18) to (2.21), and the unstable portions of the characteristic curves are shown by broken lines. In Fig. 2.2, stable solutions with amplitudes r_{10}, $k_1 r_{10}$, and frequency ω_1 are obtained for n_2/n_1 greater than the value of c. Stable solutions with r_{20}, $k_2 r_{20}$, and ω_2 are also obtained for n_2/n_1 less than d. When the value of n_2/n_1 increases, a discontinuous jump of the amplitude as well as the frequency occurs at d in the direction of the arrows. With decreasing n_2/n_1, a jump phenomenon occurs at c. Since the value of n_2/n_1 at d is greater than that at c, a hysteresis phenomenon is exhibited. For n_2/n_1 lying between c and d, there are almost periodic solutions with two frequency components ω_1 and ω_2. The state without the oscillation is unstable for any value of n_2/n_1. In Fig. 2.3, two characteristic curves are separated. No oscillation exists for n_2/n_1 between a and b. The numerical examples described above are compared with the solution obtained by using an analog computer. Figure 2.4 shows the block diagram of a computer setup for the solution of Eqs. (1.7). The symbols

---

* The coupling k and the parameters

$$\frac{n_2}{n_1} = \sqrt{\frac{L_1 C_1}{L_2 C_2}}, \qquad \left(\frac{n_2}{n_1}\right)^2 \delta = \frac{L_1}{L_2} \frac{C_1 R_1}{MS_1 - C_1 R_1}$$

are varied independently by changing the values of N, C_2, and R_2, respectively.

† One sees from Eqs. (1.3) and (2.22) that

$$\chi_1 = \sqrt{L_1/L_2}\, (n_1/n_2)^2 k, \qquad \chi_2 = \sqrt{L_2/L_1}\, (n_2/n_1)^2 k$$

To fix the values χ_1 and χ_2 it is assumed, in this paper, that $L_1 = L_2$.

in the figure follow the conventional notation. By using this setup, some representative waveforms of u(t) and v(t) are obtained in Fig. 2.5 for the parameters

$$\mu = 0.1, \; k = \chi_1 = \chi_2 = 0.5, \; (n_2/n_1)^2\delta = 0.5, \text{ and } n_2/n_1 = 1.0$$

One sees that two kinds of periodic oscillations exist and that at the higher natural frequency ω_2, u and v are approximately in phase with each other, while at the lower natural frequency ω_1, they are nearly 180° out of phase. These results show a satisfactory agreement with the assumption of the solution given by Eqs. (1.14) and (2.1), where $k_1 < 0$ and $k_2 > 0$ (see the footnote of p.4).

2.4 Self-Excited Oscillations in the Transient State

(a) Singular Points Correlated with the Periodic And Almost Periodic Solutions

In the preceding sections we have discussed the steady-state oscillations and their stability. As one sees in Fig. 2.2, two kinds of periodic oscillations are stably sustained for the values of n_2/n_1 between c and d. It depends on the initial conditions as regards which kind of oscillations occurs. This problem is made clear from the study of the solution in the transient state, which, with the lapse of time, yields ultimately a periodic solution. For this purpose it is useful to investigate the integral curves of the following equation derived from Eqs. (2.9), i.e.,

$$\frac{dR_2}{dR_1} = \frac{Y(R_1, R_2)}{X(R_1, R_2)} = -\frac{\omega_2^2 k_1}{\omega_1^2 k_2} \frac{(\rho_2 - 2R_1 - R_2)R_2}{(\rho_1 - R_1 - 2R_2)R_1} \tag{2.23}$$

Steady-state solutions are correlated with $R_1(t)$ = constant, $R_2(t)$ = constant of Eqs. (2.9), i.e., with the singular points of Eq. (2.23) for which both $X(R_1, R_2)$ and $Y(R_1, R_2)$ vanish.

The type of the singular point is classified according to the nature of characteristic roots of Eqs. (2.15). From the conditions (2.18) to (2.21), the boundaries of such a classification are given by the relations

$$\rho_1 = 0, \qquad \rho_2 = 0 \tag{2.24}$$

and

$$\rho_1 - 2\rho_2 = 0, \qquad \rho_2 - 2\rho_1 = 0 \tag{2.25}$$

By making use of Eqs. (2.8) and (1.12), Eqs. (2.24) and (2.25) may be written as

$$[2k^2 - (\frac{n_2}{n_1})^2\delta - 1](\frac{n_2}{n_1})^2 + (1 - 2k^2)(\frac{n_2}{n_1})^2\delta + 1$$
$$\pm [1 - (\frac{n_2}{n_1})^2\delta\ ] \sqrt{[1 - (\frac{n_2}{n_1})^2]^2 + 4(\frac{n_2}{n_1})^2 k^2} = 0 \tag{2.26}$$

$$2k^2(\frac{n_2}{n_1})^4 - (\frac{n_2}{n_1})^2\delta[(\frac{n_2}{n_1})^4 + 2(k^2 - 1)(\frac{n_2}{n_1})^2 + 1]$$
$$\pm 3(\frac{n_2}{n_1})^2\delta[(\frac{n_2}{n_1})^2 - 1] \sqrt{[1 - (\frac{n_2}{n_1})^2]^2 + 4(\frac{n_2}{n_1})^2 k^2} = 0 \tag{2.27}$$

It is not difficult to determine the type of the singular point according to the values k, $(n_2/n_1)^2\delta$ and n_2/n_1. Figure 2.6 and Table 2.1 show the classification of the singular points. When the values of $(n_2/n_1)^2\delta$ and n_2/n_1 are chosen inside the regions I to VI in Fig. 2.6, the types of singular points O, A, B, and C, which correspond to the steady-state solutions (1) to (4) of Eqs. (2.11), respectively, are shown in Table 2.1 (see Fig. 2.7). The singularities not shown in the table (marked by ——) means that the singular points do not exist in the region $R_1 \geq 0$ and $R_2 \geq 0$ of R_1R_2 plane.

Table 2.1 Classification of singular points in the regions of Fig. 2.6

Singularity / Region	O	A	B	C
I	Stable node	———	———	———
II	Saddle(unstable)	———	Stable node	———
III	Unstable node	Saddle(unstable)	Stable node	———
IV	Unstable node	Stable node	Stable node	Saddle(unstable)
V	Unstable node	Stable node	Saddle(unstable)	———
VI	Saddle(unstable)	Stable node	———	———

(b) Phase-Plane Analysis

It is interesting to consider the integral curves of Eq. (2.23) for a certain typical case. Figure 2.7 shows an example of the phase-plane portrait where $n_2/n_1 = 1.05$ and, as before, $k = 0.5$ and $(n_2/n_1)^2\delta = 0.5$. The stable oscillations are represented by singular points A and B, whose amplitude are given by $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$, respectively. Singular point C represents an almost periodic oscillation which has two frequency components ω_1 and ω_2. It is, however, unstable. The origin is also an unstable singularity. A representative point on the integral curves moves in the direction of the arrows with increasing time. The integral curves (thick line) that approach C divide the R_1R_2 plane into two domains of attraction, in one of which all integral curves tend to the singularity A and in the other to the singularity B. This particular integral curve is reffered to as a separatrix.

From Eqs. (1.14), (2.1), and (2.2), we obtain

$$u(0) = r_1(0) \cos \theta_1(0) + r_2(0) \cos \theta_2(0)$$

$$v(0) = k_1 r_1(0) \cos \theta_1(0) + k_2 r_2(0) \cos \theta_2(0) \tag{2.28}$$

$$\begin{aligned} \dot{u}(0) &= -\omega_1 r_1(0) \sin \theta_1(0) - \omega_2 r_2(0) \sin \theta_2(0) \\ \dot{v}(0) &= -\omega_1 k_1 r_1(0) \sin \theta_1(0) - \omega_2 k_2 r_2(0) \sin \theta_2(0) \end{aligned} \tag{2.29}$$

where $u(0)$, ..., $\theta_2(0)$ are the values of $u(t)$, ..., $\theta_2(t)$ at $t = 0$. Eliminating $\theta_1(0)$ and $\theta_2(0)$ from Eqs. (2.28) and (2.29) yields

$$\begin{aligned} R_1(0) \equiv r_1(0)^2 &= \frac{1}{(k_2 - k_1)^2} \left\{[k_2 u(0) - v(0)]^2 + \frac{1}{\omega_1^2} [k_2 \dot{u}(0) - \dot{v}(0)]^2\right\} \\ R_2(0) \equiv r_2(0)^2 &= \frac{1}{(k_2 - k_1)^2} \left\{[k_1 u(0) - v(0)]^2 + \frac{1}{\omega_2^2} [k_1 \dot{u}(0) - \dot{v}(0)]^2\right\} \end{aligned} \tag{2.30}$$

By making use of these relations, the two domains of attraction in Fig. 2.7 are reproduced in the uv plane. In Fig. 2.8, for simplicity, we considered the case where $\dot{u}(0) = \dot{v}(0) = 0$. Then the regions become symmetric about the origin. From these figures the relationship between the initial conditions and the resulting oscillations is apparent: an oscillation started with any initial conditions in the region containing the singularity A tends ultimately to the singularity A, which corresponds to the periodic oscillation with the frequency ω_1, whereas an oscillation started from the region containing the singularity B tends to the singularity B, which corresponds to the oscillation with the frequency ω_2. It is also seen that the singularity C is situated on the boundary curve between these two regions.

2.5 Concluding Remarks

Periodic and almost periodic oscillations which occur in a self-oscillatory system with two degrees of freedom have been investigated. The system is described by two second-order differential equations, one of which contains a nonlinear damping term. Assuming that a small parameter is associated with the nonlinear term, we derive an autonomous system by making use of the averaging method. When the ratio between the two natural frequencies of the system is not in the neighborhood of an integer or a fraction, two kinds of periodic oscillations, each having one of the natural frequencies, are stably sustained. On the other hand, the combination oscillation which consists of two harmonic components is unstable. The frequency and amplitude characteristics have been shown for some values of the system parameters.

The transient states of the oscillations are also studied by making use of the phase-plane analysis. From this analysis the relationship between the initial conditions and the resulting oscillations is made clear.

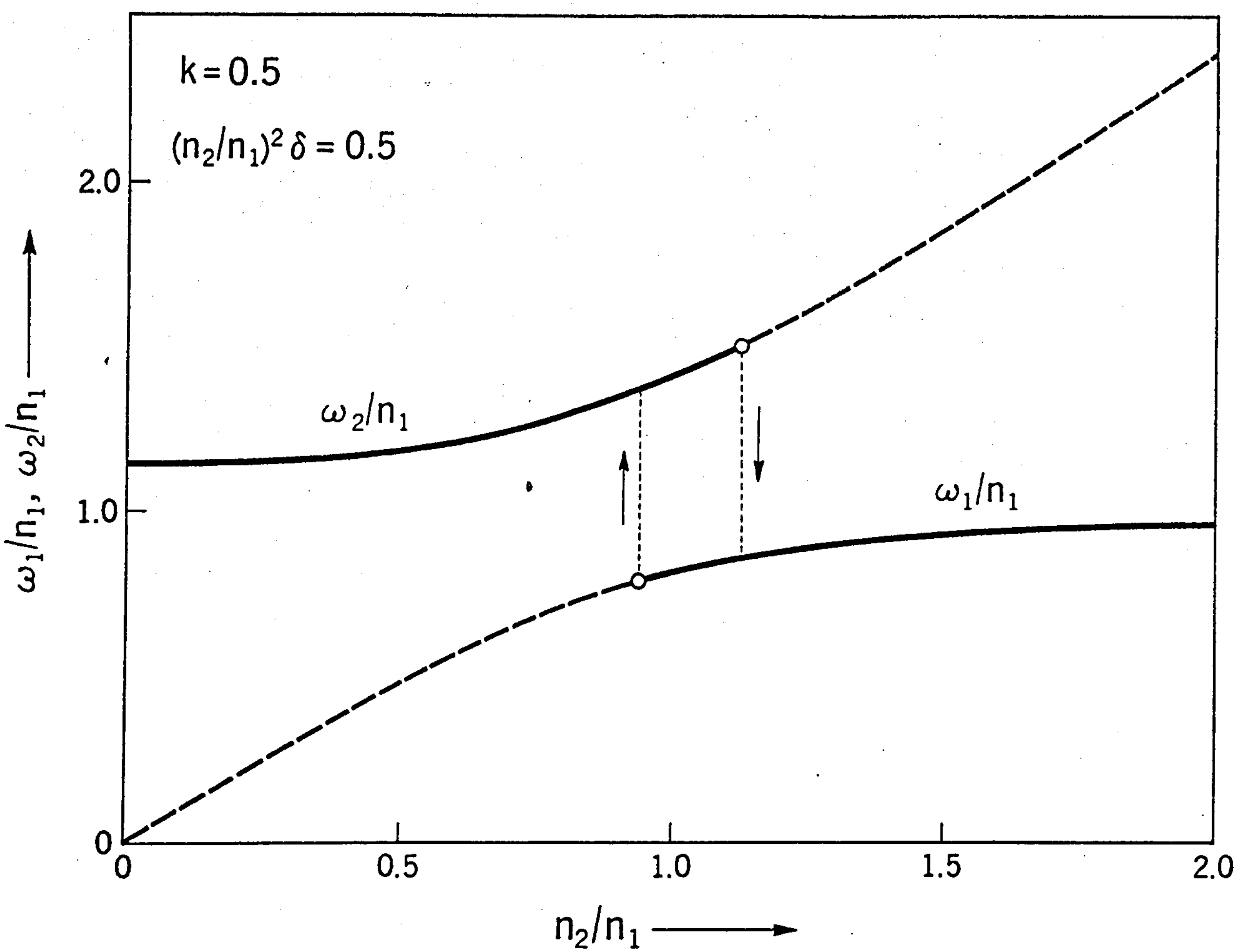


Fig. 2.1. Frequency characteristic of the self-excited oscillation.

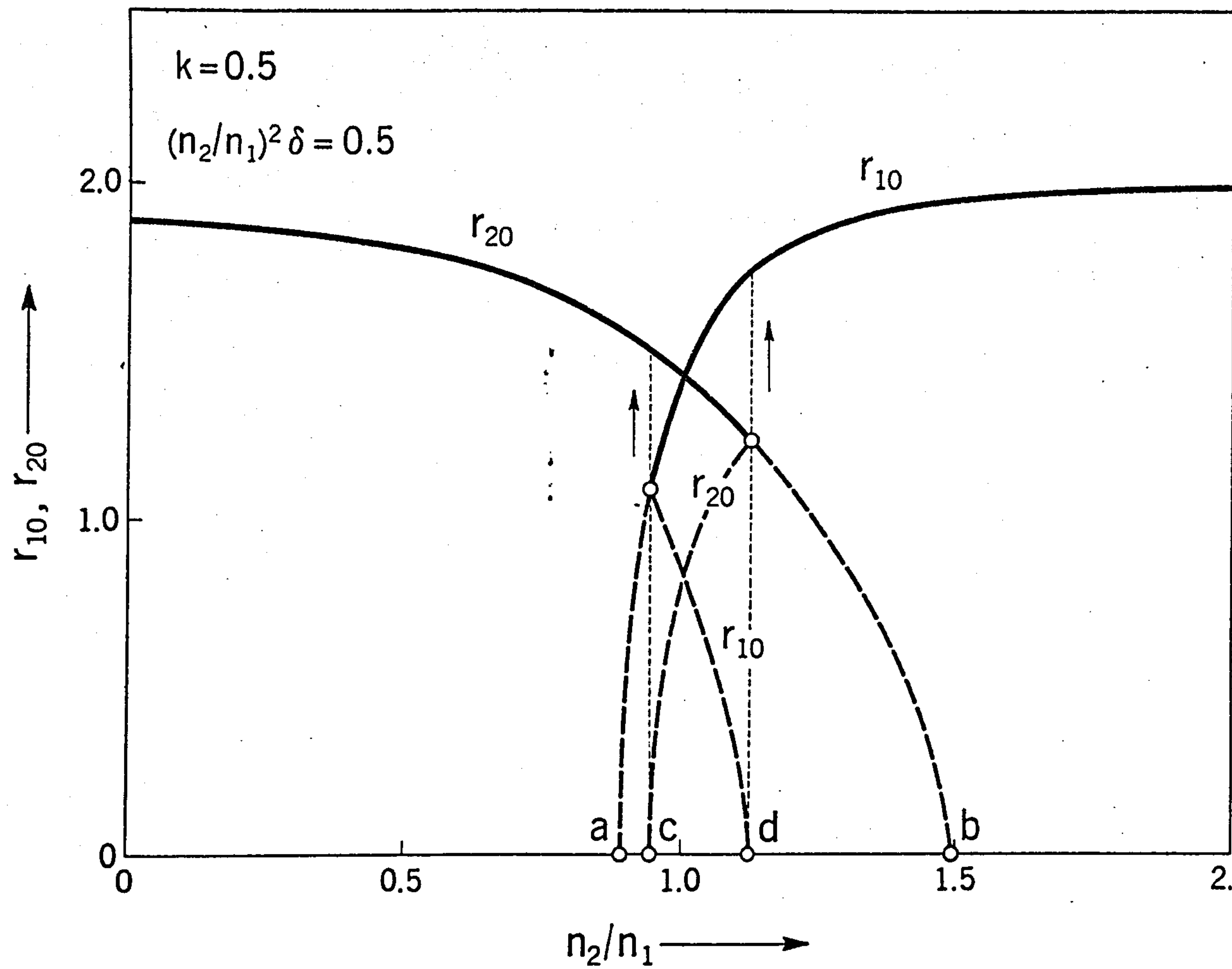


Fig. 2.2(a). Amplitude characteristic (r_{10} and r_{20}) of the self-excited oscillation.

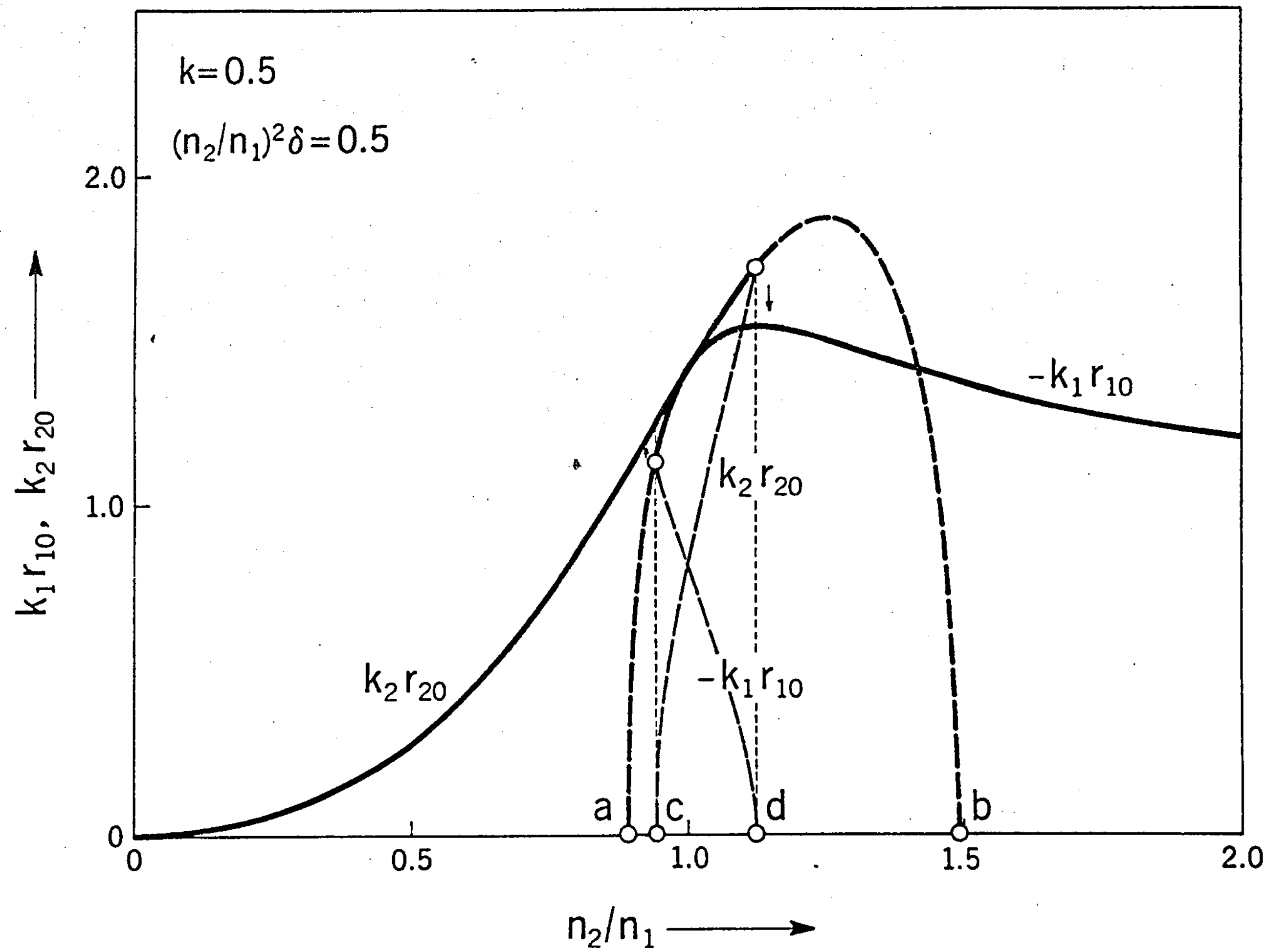


Fig. 2.2(b). Amplitude characteristic (k_1r_{10} and k_2r_{20}) of the self-excited oscillation.

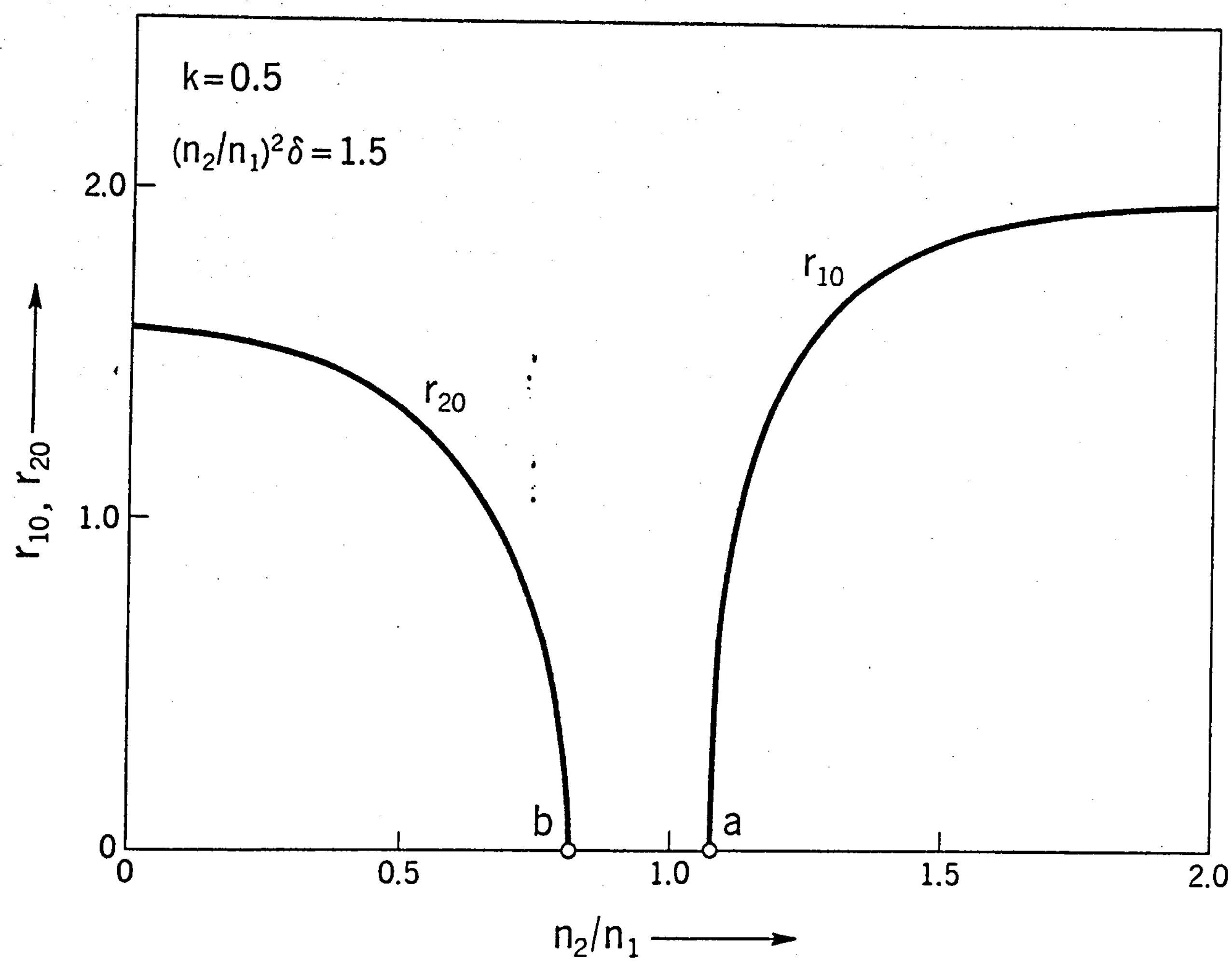


Fig. 2.3(a). Amplitude characteristic (r_{10} and r_{20}) of the self-excited oscillation.

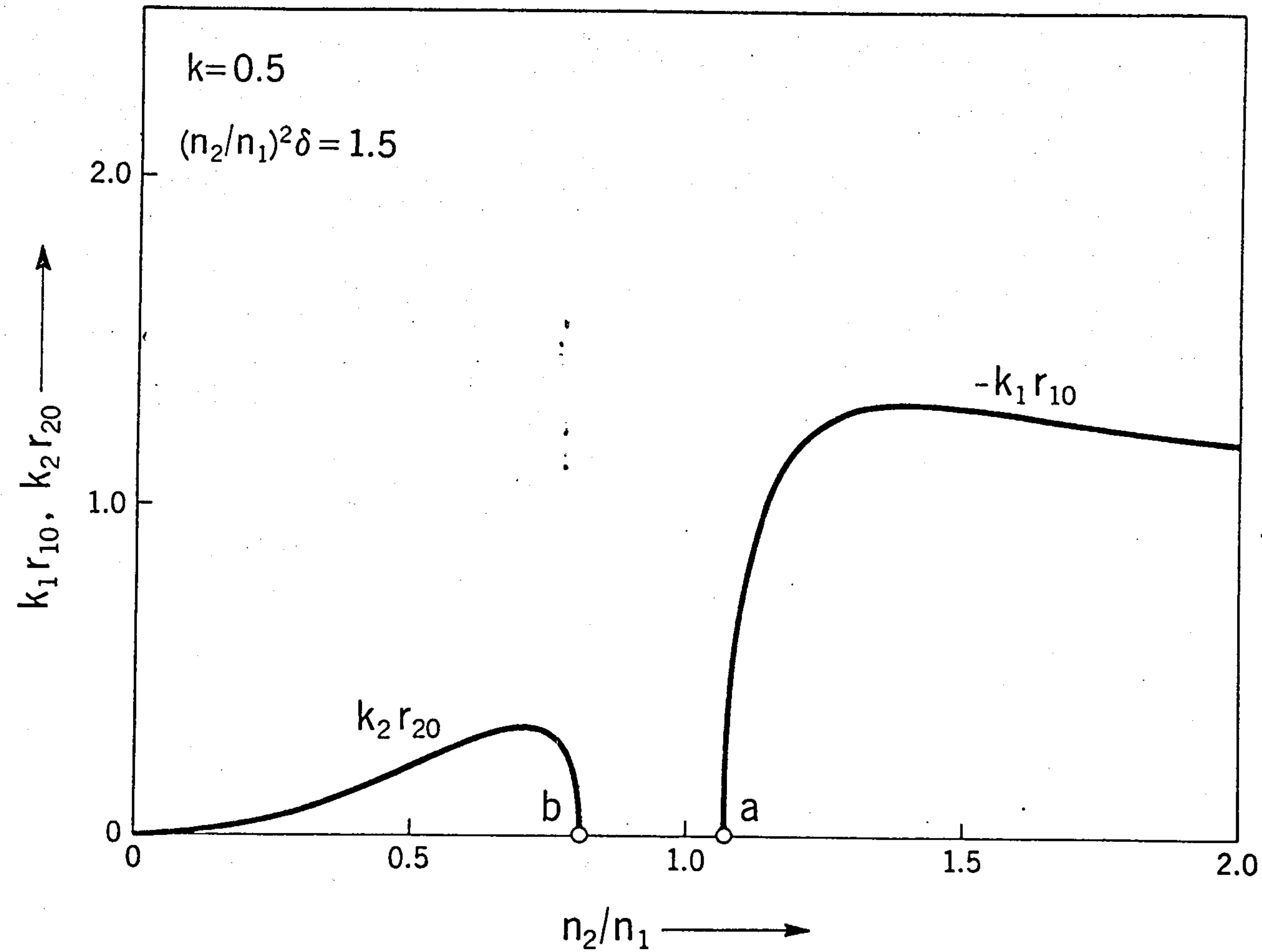


Fig. 2.3(b). Amplitude characteristic (k_1r_{10} and k_2r_{20}) of the self-excited oscillation.

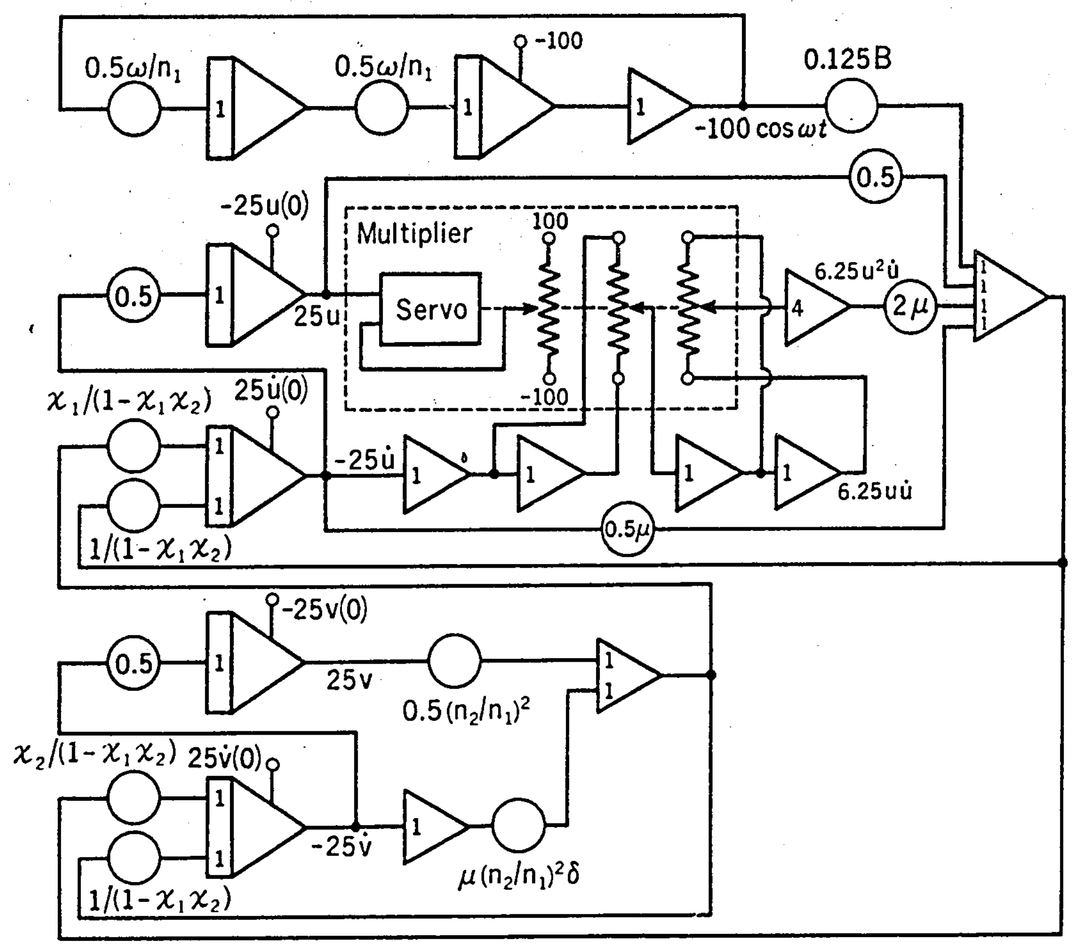
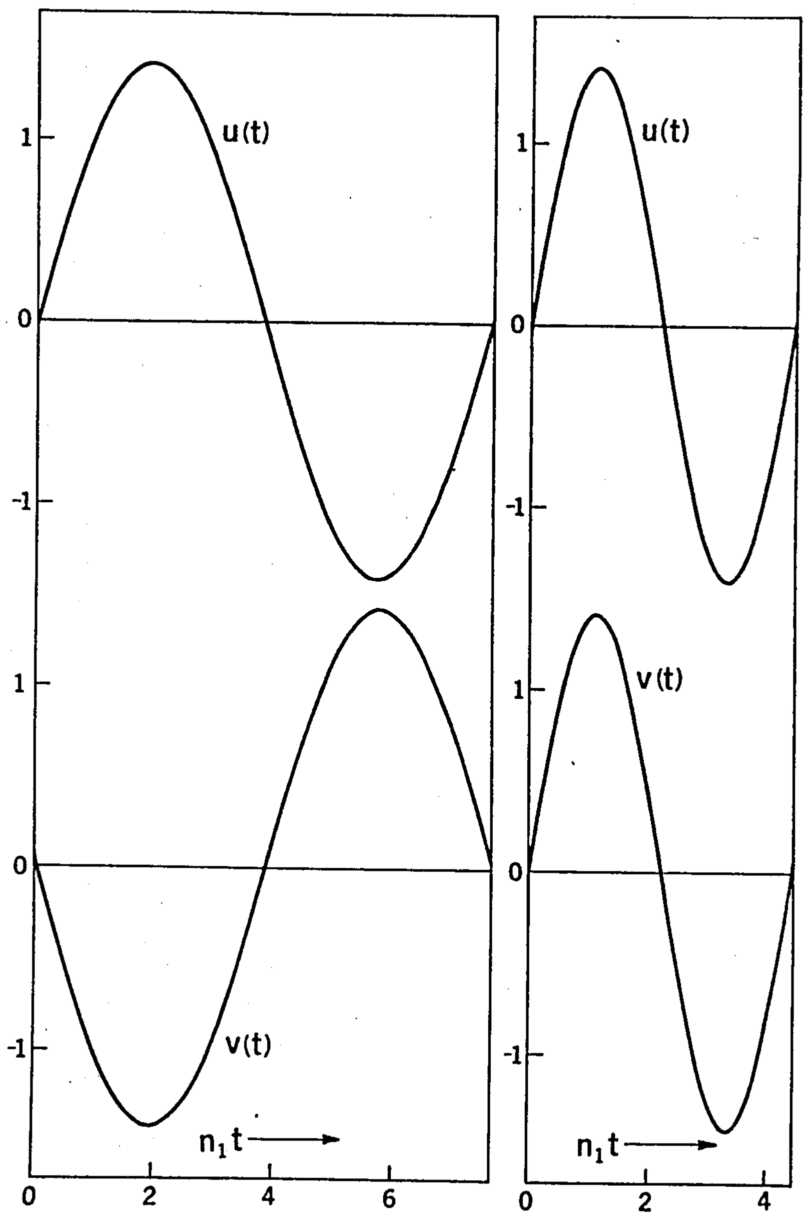


Fig. 2.4. Computer block diagram for Eqs. (1.7).



(a) ω_1 (b) ω_2

Fig. 2.5. Waveforms of two types of the self-excited oscillations [$\mu = 0.1$, $k = 0.5$, $(n_2/n_1)^2\delta = 0.5$, $n_2/n_1 = 1.0$].

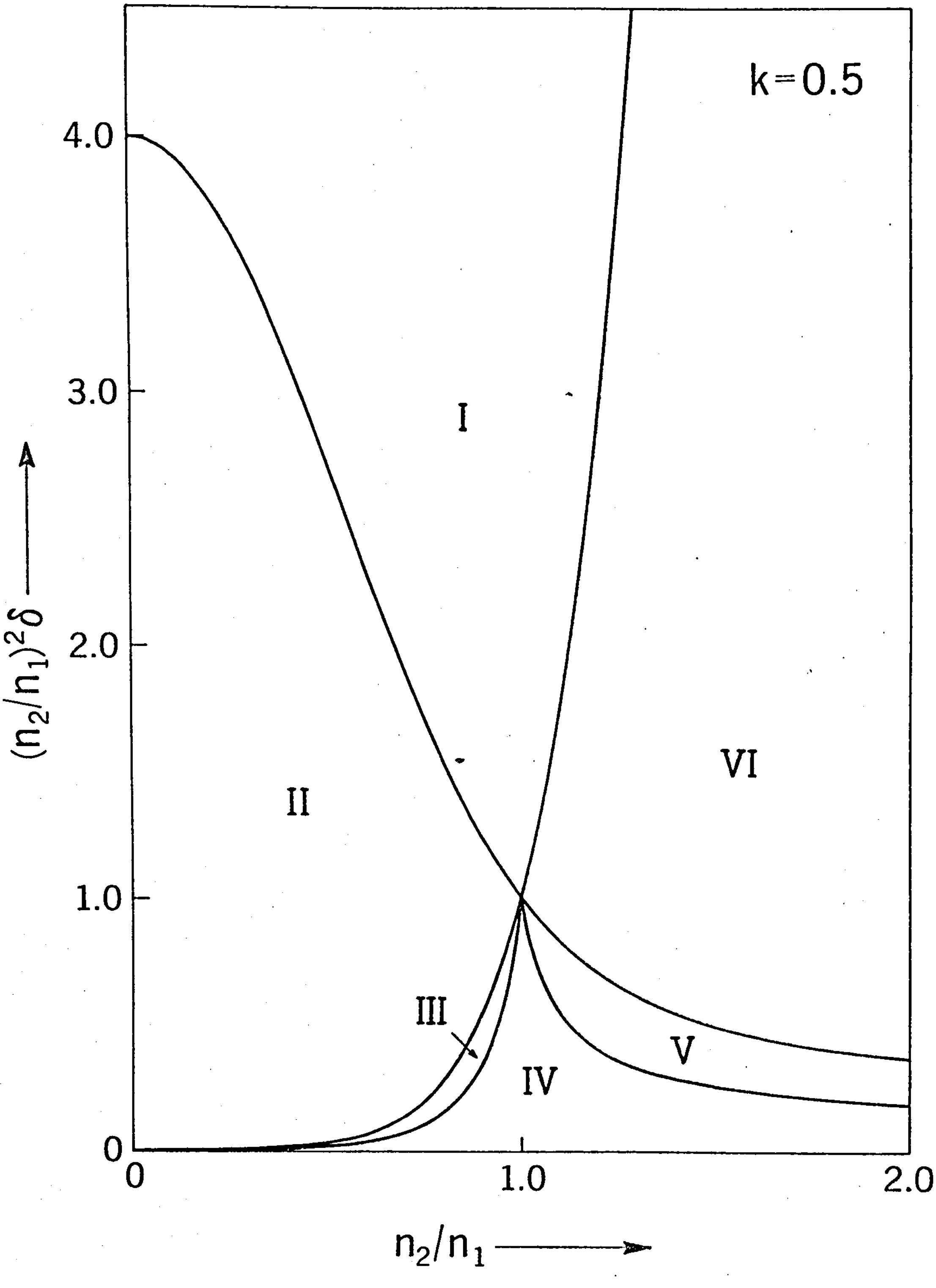


Fig. 2.6. Classification of singular points.

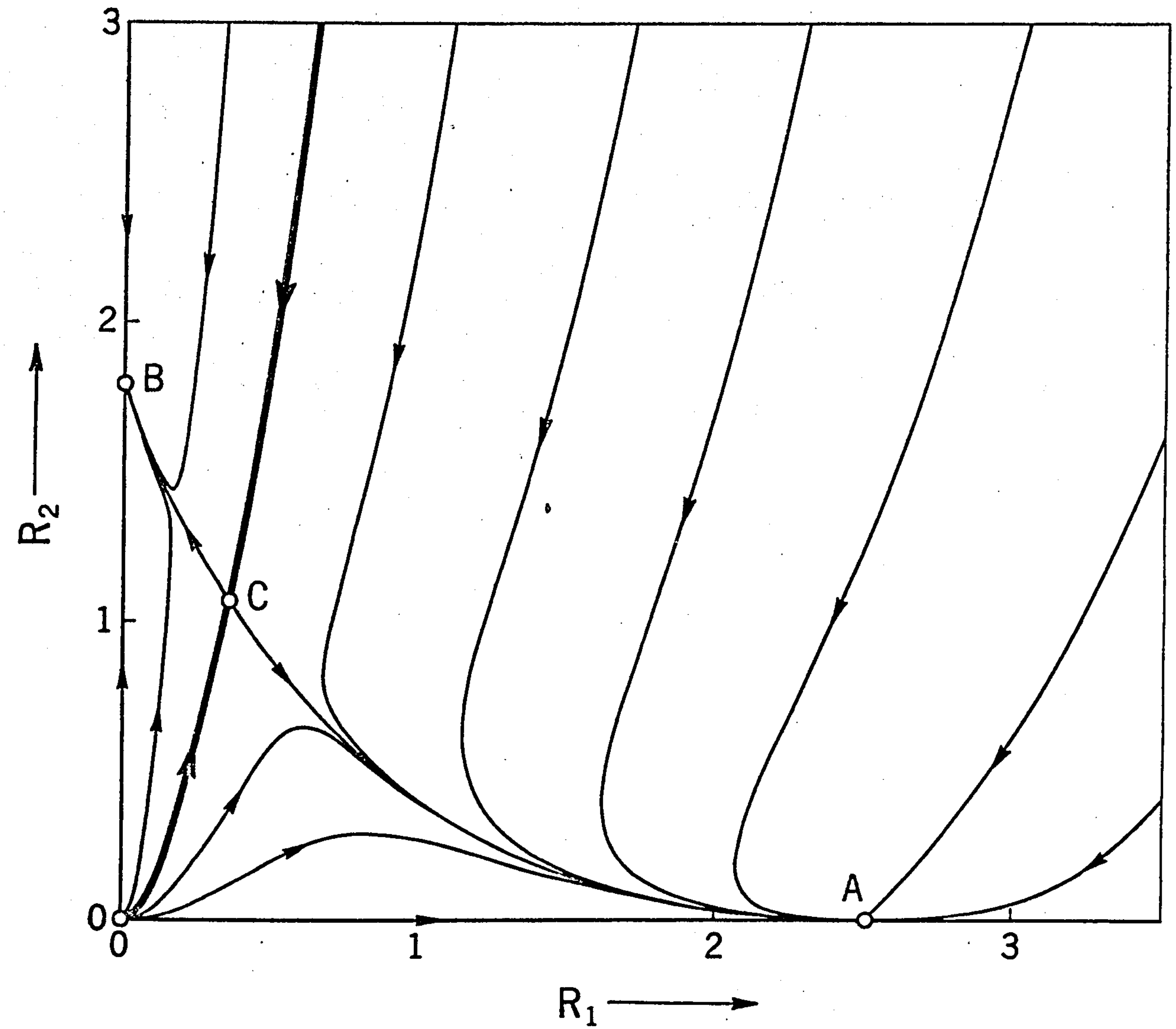


Fig. 2.7. Phase-plane portrait of Eq. (2.23), the system parameters being $k = 0.5$, $(n_2/n_1)^2\delta = 0.5$, and $n_2/n_1 = 1.05$.

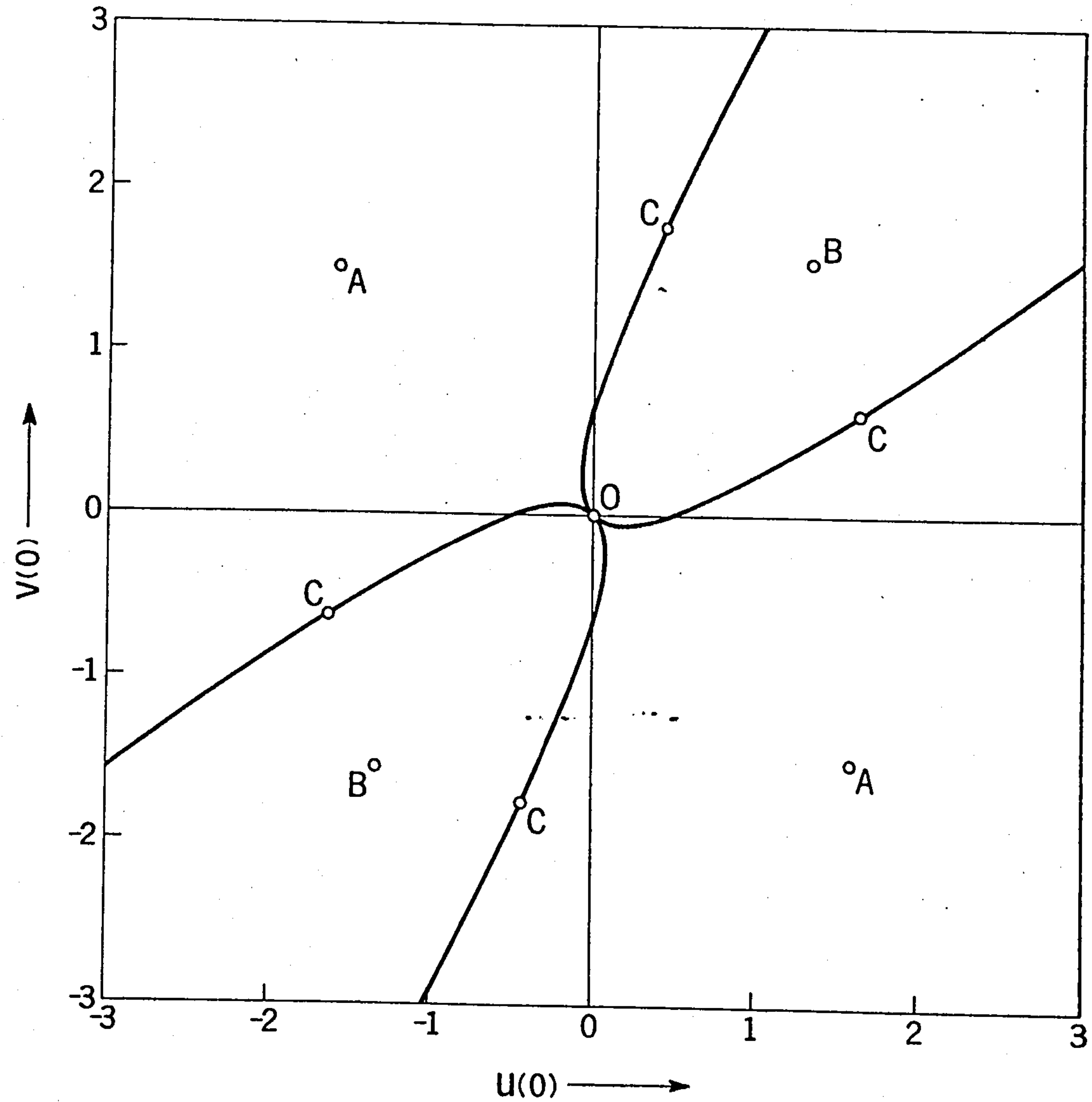


Fig. 2.8. Regions of initial conditions in the uv plane leading to the two types of periodic oscillations [$\dot{u}(0) = 0$ and $\dot{v}(0) = 0$].

CHAPTER 3

INTERNAL RESONANCE IN A SELF-OSCILLATORY

SYSTEM WITH TWO DEGREES OF FREEDOM

3.1 Introduction

In the preceding chapter, we treated the self-excited oscillations in a system with two degrees of freedom. It was shown that two types of periodic oscillations having the natural frequency ω_1 and ω_2, respectively, are stably sustained. However, when we derived the autonomous system by using the averaging method, we neglected the case in which ω_2 is equal to ω_1 or $3\omega_1$. When such relationships were satisfied between ω_1 and ω_2, we noticed that additional terms appeared in the averaged equations. This fact suggests us that some special phenomena may occur when the ratio of the two natural frequencies is in the neighborhood of unity or 1/3. In this chapter we discuss the self-excited oscillations under these conditions.

It is well known that the phenomenon of frequency entrainment occurs when a periodic driving force is applied to a self-oscillatory system with one degree of freedom [6, 10, 26, 27, 35]. The self-oscillatory system under consideration has two natural frequencies. Hence we may expect that the phenomenon of frequency entrainment between the two natural frequencies, i.e., the internal resonance, is likely to occur when the ratio between them is in the neighborhood of unity or 1/3.

It is assumed that the frequency of the entrained oscillations is not equal to either of the two natural frequencies, but is in the neighborhood of them. Autonomous systems are derived by using the averaging method. The steady-state oscillations and their stability are discussed.

3.2 Internal Resonance Which Occurs When $3\omega_1 \cong \omega_2$.

(a) Derivation of an Autonomous System by Using the Averaging Method

We consider the standard form of differential equations (1.15), i.e.,

$$\begin{aligned} \ddot{x} + \omega_1^2 x &= \frac{\mu}{n_1} \frac{\omega_1^2}{k_2 - k_1} \left\{k_2[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y})\right\} \\ &\equiv \mu F_1(x, y, \dot{x}, \dot{y}) \\ \ddot{y} + \omega_2^2 y &= \frac{\mu}{n_1} \frac{\omega_2^2}{k_1 - k_2} \left\{k_1[1 - (x + y)^2](\dot{x} + \dot{y}) + \delta(k_1\dot{x} + k_2\dot{y})\right\} \\ &\equiv \mu G_1(x, y, \dot{x}, \dot{y}) \end{aligned} \tag{3.1}$$

As mentioned in Chapter 2, it is expected that the entrainment of frequency occurs when $3\omega_1$ is in the neighborhood of ω_2. To investigate this phenomenon, we make use of the expansion

$$\begin{aligned} \omega_1 &= \omega_{10} + \mu\omega_{11} + \cdots \\ \omega_2 &= \omega_{20} + \mu\omega_{21} + \cdots \end{aligned} \tag{3.2}$$

The entrained oscillation consists of two components of frequencies ω_{10} and ω_{20}, which are in the neighborhood of ω_1 and ω_2, respectively, and are related by $3\omega_{10} = \omega_{20}$, as one sees afterwards.

Substituting Eqs. (3.2) into Eqs. (3.1) yields

$$\begin{aligned} \ddot{x} + \omega_{10}^2 x &= \mu F_{10}(x, y, \dot{x}, \dot{y}) + O(\mu^2) \\ \ddot{y} + \omega_{20}^2 y &= \mu G_{10}(x, y, \dot{x}, \dot{y}) + O(\mu^2) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} F_{10}(x, y, \dot{x}, \dot{y}) &= F_1(x, y, \dot{x}, \dot{y}) - 2\omega_{10}\omega_{11}x \\ G_{10}(x, y, \dot{x}, \dot{y}) &= G_1(x, y, \dot{x}, \dot{y}) - 2\omega_{20}\omega_{21}y \end{aligned} \tag{3.4}$$

We assume that the solution of Eqs. (3.3) takes the form

$$
\begin{aligned}
x(t) &= r_1(t)\cos[\omega_{10}t + \theta_1(t)] \\
y(t) &= r_2(t)\cos[\omega_{20}t + \theta_2(t)] \\
\dot{x}(t) &= -\omega_{10}r_1(t)\sin[\omega_{10}t + \theta_1(t)] \\
\dot{y}(t) &= -\omega_{20}r_2(t)\sin[\omega_{20}t + \theta_2(t)]
\end{aligned}
\tag{3.5}
$$

Hence the solution of Eqs. (1.9) is written as

$$
\begin{aligned}
u(t) &= r_1(t)\cos[\omega_{10}t + \theta_1(t)] + r_2(t)\cos[\omega_{20}t + \theta_2(t)] \\
v(t) &= k_1r_1(t)\cos[\omega_{10}t + \theta_1(t)] + k_2r_2(t)\cos[\omega_{20}t + \theta_2(t)]
\end{aligned}
\tag{3.6}
$$

Substituting Eqs. (3.5) into Eqs. (3.3) and using the averaging method as we have done in Chap. 2, we obtain

$$
\begin{aligned}
\dot{r}_1 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_{10}} f_{10}(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_{10}t + \theta_1)dt \\
\dot{r}_2 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_{20}} g_{10}(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_{20}t + \theta_2)dt \\
r_1\dot{\theta}_1 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_{10}} f_{10}(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_{10}t + \theta_1)dt \\
r_2\dot{\theta}_2 &= -\lim_{T\to\infty}\frac{\mu}{T}\int_0^T \frac{1}{\omega_{20}} g_{10}(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_{20}t + \theta_2)dt
\end{aligned}
\tag{3.7}
$$

where f_{10} and g_{10} in the right-hand sides of Eqs. (3.7) are obtained by substitution of Eqs. (3.5) into Eqs. (3.4) (see Appendix I). The functions f_{10} and g_{10} in Eqs. (3.7) contain terms of frequencies ω_{10}, ω_{20}, $3\omega_{10}$, $3\omega_{20}$, $\omega_{20} \pm 2\omega_{10}$ and $2\omega_{20} \pm \omega_{10}$. Therefore, if there exists the relationship $\omega_{10} = \omega_{20}$ or $3\omega_{10} = \omega_{20}$, some of these frequencies which apparently look different from ω_{10} or ω_{20} coincide with them. The former case will be discussed in the next section.

Performing the integrations of Eqs. (3.7) under the condition $3\omega_{10} = \omega_{20}$ yields an autonomous system.

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3} r_1^3 \cos(3\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} r_1^2 r_2 \sin(3\theta_1 - \theta_2)] \tag{3.8}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2} r_1^3 \sin(3\theta_1 - \theta_2)]$$

(b) Steady-State Solutions

The steady-state solutions of Eqs. (3.8) are obtained by equating $\dot{r}_1 = \dot{r}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$. Denoting the steady-state values of these variables by r_{10}, r_{20}, θ_{10}, and θ_{20}, respectively, we obtain

$$(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10} - r_{10}^2 r_{20} \cos(3\theta_{10} - \theta_{20}) = 0$$

$$(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20} - \frac{1}{3} r_{10}^3 \cos(3\theta_{10} - \theta_{20}) = 0$$

$$\omega_{11}r_{10} + m_1 r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) = 0 \tag{3.9}$$

$$\omega_{21}r_{20} - \frac{1}{3} m_2 r_{10}^3 \sin(3\theta_{10} - \theta_{20}) = 0$$

where

$$m_1 = \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}, \qquad m_2 = \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2} \tag{3.10}$$

We see, from Eqs. (3.9), that there are three different states of equilibrium i.e.,

$$
\begin{aligned}
&(1) \quad r_{10} = 0, \quad r_{20} = 0 \\
&(2) \quad r_{10} = 0, \quad r_{20} = \sqrt{\rho_2}, \quad \omega_{21} = 0 \\
&(3) \quad r_{10} \neq 0, \quad r_{20} \neq 0
\end{aligned}
\tag{3.11}
$$

The first and second cases are identical with those already discussed in (1) and (3) of Eqs. (2.11), respectively. We are particularly interested in the third case where the two frequency components are not zero and are entrained mutually by the relation $3\omega_{10} = \omega_{20}$. As will be mentioned later, this state of equilibrium is stable. Thus the internal resonance occurs. We see from Eqs. (3.2) that

$$
\begin{aligned}
3\omega_1 - \omega_2 &= 3\omega_{10} + 3\mu\omega_{11} - \omega_{20} - \mu\omega_{21} + O(\mu^2) \\
&= \mu(3\omega_{11} - \omega_{21}) + O(\mu^2)
\end{aligned}
\tag{3.12}
$$

Eliminating $3\theta_{10} - \theta_{20}$ from Eqs. (3.9) gives us

$$
\begin{aligned}
&(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10}^2 = 3(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20}^2 \\
&\left\{[3m_1(\rho_1 - r_{10}^2 - 2r_{20}^2) + m_2(\rho_2 - r_{10}^2 - r_{20}^2)]^2 + (3\omega_{11} - \omega_{21})^2\right\}r_{20}^2 \\
&\qquad = (3m_1 r_{20}^2 + \frac{m_2}{3} r_{10}^2)^2 r_{10}^2
\end{aligned}
\tag{3.13}
$$

Solving Eqs. (3.12) and (3.13) simultaneously gives the amplitudes r_{10} and r_{20}. The phase difference $3\theta_{10} - \theta_{20}$, and the frequencies ω_{11}, ω_{21} may also be found to be*

$$
\sin(3\theta_{10} - \theta_{20}) = -\frac{r_{20}}{r_{10}} \frac{3\omega_{11} - \omega_{21}}{3m_1 r_{20}^2 + \frac{1}{3} m_2 r_{10}^2}
$$

---

* The phase angles θ_{10} and θ_{20} are not determined, but only the phase difference $3\theta_{10} - \theta_{20}$ is fixed. This result is natural since the fundamental equations (1.9) are autonomous ones.

$$\cos(3\theta_1 - \theta_{20}) = \frac{1}{r_{10}r_{20}}(\rho_1 - r_{10}^2 - 2r_{20}^2)$$

$$= \frac{3r_{20}}{r_{10}^3}(\rho_2 - 2r_{10}^2 - r_{20}^2) \tag{3.14}$$

$$\omega_{11} = m_1 r_{10} r_{20} \sin(3\theta_{10} - \theta_{20})$$

$$\omega_{21} = -\frac{m_2 r_{10}^3}{3r_{20}} \sin(3\theta_{10} - \theta_{20}).$$

Hence, from Eqs. (3.2), the entrained frequencies ω_{10} and ω_{20} are determined.

(c) Stability Investigation

The stability of the equilibrium states as given by Eqs. (3.11) is studied by solving the variational equations derived from Eqs. (3.8). By a procedure like that used in Sec. 2.3b, we consider the small variations ξ_1, ξ_2, η_1, and η_2 from the equilibrium state defined by

$$r_1 = r_{10} + \xi_1, \qquad r_2 = r_{20} + \xi_2$$
$$\theta_1 = \theta_{10} + \eta_1, \qquad \theta_2 = \theta_{20} + \eta_2 \tag{3.15}$$

Substituting Eqs. (3.15) into Eqs. (3.8) and neglecting terms of higher degree than the first in ξ and η, we obtain the variational equations, i.e.,

$$\begin{aligned}
\dot{\xi}_1 &= a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\eta_1 + a_{14}\eta_2 \\
\dot{\xi}_2 &= a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\eta_1 + a_{24}\eta_2 \\
\dot{\eta}_1 &= a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\eta_1 + a_{34}\eta_2 \\
\dot{\eta}_2 &= a_{41}\xi_1 + a_{42}\xi_2 + a_{43}\eta_1 + a_{44}\eta_2
\end{aligned} \tag{3.16}$$

where

$$a_{11} = \mu m_1[\rho_1 - 3r_{10}^2 - 2r_{20}^2 - 2r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})]$$

$$
\begin{aligned}
a_{12} &= -\mu m_1[4r_{10}r_{20} + r_{10}^2 \cos(3\theta_{10} - \theta_{20})] \\
a_{13} &= 3\mu m_1 r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) \\
a_{14} &= -\frac{1}{3} a_{13} \\
a_{21} &= -\mu m_2[4r_{10}r_{20} + r_{10}^2 \cos(3\theta_{10} - \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= \mu m_2 r_{10}^3 \sin(3\theta_{10} - \theta_{20}) \\
a_{24} &= -\frac{1}{3} a_{23} \\
a_{31} &= \mu m_1 r_{20} \sin(3\theta_{10} - \theta_{20}) \\
a_{32} &= \mu m_1 r_{10} \sin(3\theta_{10} - \theta_{20}) \\
a_{33} &= 3\mu m_1 r_{10} r_{20} \cos(3\theta_{10} - \theta_{20}) \\
a_{34} &= -\frac{1}{3} a_{33} \\
a_{41} &= -\mu m_2 \frac{r_{10}^2}{r_{20}} \sin(3\theta_{10} - \theta_{20}) \\
a_{42} &= \frac{1}{3}\mu m_2 \frac{r_{10}^3}{r_{20}^2} \sin(3\theta_{10} - \theta_{20}) \\
a_{43} &= -\mu m_2 \frac{r_{10}^3}{r_{20}} \cos(3\theta_{10} - \theta_{20}) \\
a_{44} &= -\frac{1}{3} a_{43}
\end{aligned}
\tag{3.17}
$$

The characteristic equation of the system (3.16) is given by

$$
\begin{vmatrix}
a_{11} - \lambda & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} - \lambda & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} - \lambda & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} - \lambda
\end{vmatrix} = 0
$$

or

$$\lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + s = 0 \tag{3.18}$$

where

$$p = -(a_{11} + a_{22} + a_{33} + a_{44})$$

$$q = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

$$r = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{3.19}$$

$$s = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Since there exist the relations $a_{i3} = -3a_{i4}$ $(i = 1, \ldots, 4)$ in Eqs. (3.17), $s = 0$ in Eqs. (3.19). Therefore the characteristic equation (3.16) is reduced to

$$\lambda(\lambda^3 + p\lambda^2 + q\lambda + r) = 0 \tag{3.20}$$

i.e., one of the characteristic roots is zero.* Then, the stability conditions are given by the Routh-Hurwitz criterion (see Appendix II), i.e.,

$$p > 0, \qquad r > 0, \qquad pq - r > 0 \tag{3.21}$$

The stability conditions for the first and the second cases of Eqs. (3.11) are

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* This result corresponds to the fact that the fundamental equations (3.1) are autonomous ones and that not the phase angle θ_i but the phase difference $3\theta_{10} - \theta_{20}$ is determined by Eqs. (3.14).

also identical with the conditions (2.18) and (2.20), respectively. The stability of the entrained oscillation in which both r_{10} and r_{20} are not zero is tested by making use of the conditions (3.21), and this type of oscillation is found to be stable.

(d) Numerical Examples

As mentioned in Chap. 1 [see Eqs. (1.12)], the natural frequencies ω_i/n_1 (i = 1, 2) are obtained as a function of the coupling k (= $\sqrt{\chi_1\chi_2}$) and n_2/n_1. We now seek the relationship which k and n_2/n_1 satisfy when

$$s\omega_1 = \omega_2 \tag{3.22}$$

where s is an integer. Since ω_1^2 and ω_2^2 are the roots of the quadratic equation (1.12), we obtain

$$\left.\begin{aligned} \omega_1^2 + \omega_2^2 &= \frac{n_1^2 + n_2^2}{1 - k^2} \\ \omega_1^2\omega_2^2 &= \frac{n_1^2 n_2^2}{1 - k^2} \end{aligned}\right\} \tag{3.23}$$

Eliminating ω_1^2 and ω_2^2 from Eqs. (3.22) and (3.23) yields

$$k^2 = 1 - \frac{s^2}{(1 + s^2)^2} \frac{(n_1^2 + n_2^2)^2}{n_1^2 n_2^2} \tag{3.24}$$

By setting $s = 1$ in Eq. (3.24), we obtain $k = 0$ ($N = 0$) and $n_1 = n_2$. This fact has already been pointed out in Sec. 1.3. When $s = 3$, Eq. (3.24) becomes

$$k^2 = 1 - \frac{9}{100} \frac{[1 + (n_2/n_1)^2]^2}{(n_2/n_1)^2} \tag{3.25}$$

Figure 3.1 shows the relationship between k and n_2/n_1 given by Eq. (3.25). One may expect the occurrence of the internal resonance if k and n_2/n_1 take the values near the curve in Fig. 3.1. Under this condition, the amplitudes r_{10} and r_{20} of the entrained oscillation are calculated from Eqs. (3.13). Therefore the actual existence of the internal resonance depends on the values of ρ_1 and ρ_2 as well as $3\omega_{11} - \omega_{21}$. When $\rho_1 = 0$ or $\rho_2 = 0$, Eq. (2.26) is satisfied. This relationship between k and n_2/n_1 is also illustrated in Fig. 3.1 by broken lines for $(n_2/n_1)^2\delta = 0.5$.* Let us consider a case where

$$\mu = 0.1, \qquad k = 0.5, \qquad (n_2/n_1)^2\tilde{\delta} = 0.5$$

When $k = 0.5$, we find, from the result in Fig. 3.1, that $\omega_2 = 3\omega_1$ provided

$$n_2/n_1 \cong 0.403 \quad \text{and} \quad n_2/n_1 \cong 2.48$$

By using Eqs. (3.12) and (3.13), the amplitudes r_{10}, r_{20}, and the entrained frequency ω_{10} are calculated. Real values of the amplitudes are obtained only for $n_2/n_1 \simeq 2.48$.† Figure 3.2a shows the entrained frequency ω_{10}/n_1 as n_2/n_1 varies. For comparison's sake, ω_1/n_1 curve calculated from Eq. (1.12) is also shown in the figure. Figures 3.2b and 3.2c show the amplitude characteristics of the entrained oscillation.** We see that, when the entrainment occurs, the amplitude $k_2 r_{20}$ of the third harmonic component of the solution v(t) increases predominantly. The phase characteristic of the entrained oscillation is shown in Fig. 3.2d.

These results show that as n_2/n_1 takes the value far from 2.48, i.e., as

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* As one sees from Eqs. (2.11), these lines are the boundaries of existence of periodic oscillations, when no internal resonance occurs.

† When $n_2/n_1 = 0.403$, one sees from Fig. 3.1 that $\rho_1 < 0$. For this reason, Eqs. (3.12) and (3.13) have no real roots.

** To fix the values of χ_1 and χ_2, it is chosen that $L_1 = L_2$ (see the footnote of p.16).

the difference $|3\omega_1 - \omega_2|$ becomes larger, the amplitudes r_{20} and $k_2 r_{20}$ of the third harmonic components decrease and the amplitudes r_{10} and $k_1 r_{10}$ of the harmonic components tend to the values which are obtained in the case where no internal resonance occurs (see Fig. 2.2). The entrained frequency ω_{10} also tends to ω_1.

Next, we consider another case where

$$\mu = 0.2, \quad k = 0.8, \quad (n_2/n_1)^2\delta = 0.5$$

In this case, from Eq. (3.25) we find that $3\omega_1 = \omega_2$ provided

$$n_2/n_1 = 1.0$$

The frequency, amplitudes, and phase difference of the entrained oscillations are shown in Fig. 3.3. The dotted portions of the characteristic curves represent unstable states. One sees in Fig. 3.3 that, when n_2/n_1 is in the neighborhood of unity, there exist three kinds of entrained oscillations, one of which is stable. It is to be mentioned that the other kind of periodic oscillation of the frequency ω_2 also exists. When two kinds of periodic oscillations exist, it depends on the initial condition as regards which kind of oscillations occurs. Figure 3.4 shows the time-response curves of the entrained oscillation which is obtained by analog-computer analysis (see Fig. 2.4).

3.3 Internal Resonance Which Occurs When $\omega_1 \cong \omega_2$

(a) Derivation of an Autonomous System by Using the Averaging Method

The entrainment of frequency also occurs when $\omega_1 \cong \omega_2$. As mentioned in Sec. 1.3, this type of internal resonance occurs when $n_1 \cong n_2$ and $k \cong 0$ (i.e., $N \cong 0$). In this case the transformation of the differential equations to the standard form ceases to be meaningful. Therefore, we consider the fundamental equations

(1.9). Let the entrained frequency be ω_0 which is in the neighborhood of ω_1 and ω_2, i.e., n_1 and n_2. Then the solution of Eqs. (1.9) is written as

$$
\begin{aligned}
u(t) &= r_u(t) \cos [\omega_0 t + \theta_u(t)] \\
v(t) &= r_v(t) \cos [\omega_0 t + \theta_v(t)] \\
\dot{u}(t) &= -\omega_0 r_u(t) \sin [\omega_0 t + \theta_u(t)] \\
\dot{v}(t) &= -\omega_0 r_v(t) \sin [\omega_0 t + \theta_v(t)]
\end{aligned}
\tag{3.26}
$$

Substituting Eqs. (3.26) into Eqs. (1.9) and applying the averaging method as before, we obtain the autonomous system

$$
\begin{aligned}
\dot{r}_u &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} [\mu\omega_0 n_1 (1 - \tfrac{1}{4} r_u^2) r_u + \chi_1 n_2^2 r_v \sin (\theta_u - \theta_v) \\
&\quad - \mu\omega_0 \chi_1 \frac{n_2^2}{n_1} \delta r_v \cos (\theta_u - \theta_v)] \\
\dot{r}_v &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} [-\mu\omega_0 \frac{n_2^2}{n_1} \delta r_v - \chi_2 n_1^2 r_u \sin (\theta_u - \theta_v) \\
&\quad + \mu\omega_0 \chi_2 n_1 (1 - \tfrac{1}{4} r_u^2) r_u \cos (\theta_u - \theta_v)] \\
r_u\dot{\theta}_u &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} \{[n_1^2 - (1 - \chi_1\chi_2)\omega_0^2] r_u \\
&\quad + \mu\omega_0 \chi_1 \frac{n_2^2}{n_1} \delta r_v \sin (\theta_u - \theta_v) + \chi_1 n_2^2 r_v \cos (\theta_u - \theta_v)\} \\
r_v\dot{\theta}_v &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} \{[n_2^2 - (1 - \chi_1\chi_2)\omega_0^2] r_v \\
&\quad + \mu\omega_0 \chi_2 n_1 (1 - \tfrac{1}{4} r_u^2) r_u \sin (\theta_u - \theta_v) + \chi_2 n_1^2 r_u \cos (\theta_u - \theta_v)\}
\end{aligned}
\tag{3.27}
$$

(b) Steady-State Solutions

The steady-state solutions of Eqs. (3.27) are obtained by equating $\dot{r}_u = \dot{r}_v = \dot{\theta}_u = \dot{\theta}_v = 0$. After some algebraic manipulation, we obtain

$$
\begin{aligned}
r_{u0}^2 &= 4\left(1 - \frac{\omega_0^2 - n_1^2}{\omega_0^2 - n_2^2}\frac{n_2^2}{n_1^2}\delta\right) \\
r_{v0}^2 &= \frac{\chi_2}{\chi_1}\frac{\omega_0^2 - n_1^2}{\omega_0^2 - n_2^2} r_{u0}^2 \\
\sin(\theta_{u0} - \theta_{v0}) &= -\frac{\mu}{\chi_1\omega_0}\frac{\omega_0^2 - n_1^2}{\omega_0^2 - n_2^2}\frac{n_2^2}{n_1}\delta\frac{r_{u0}}{r_{v0}} \\
\cos(\theta_{u0} - \theta_{v0}) &= \frac{\omega_0^2 - n_1^2}{\chi_1\omega_0^2}\frac{r_{u0}}{r_{v0}}
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
&[(1 - \chi_1\chi_2)\omega_0^4 - (n_1^2 + n_2^2)\omega_0^2 + n_1^2n_2^2](\omega_0^2 - n_2^2) \\
&\quad + (\mu\frac{n_2^2}{n_1}\delta)^2(\omega_0^2 - n_1^2)\omega_0^2 = 0
\end{aligned}
\tag{3.29}
$$

Equation (3.29) determines the entrained frequency ω_0. Substituting this value into Eqs. (3.28) gives the amplitudes r_{u0}, r_{v0}, and the phase difference $\theta_{u0} - \theta_{v0}$ of the entrained oscillation. It is noted that the trivial solution $r_{u0} = r_{v0} = 0$ of Eqs. (3.27) always exists.

When the amplitude r_{u0} of the entrained oscillation is equal to zero, we obtain from Eqs. (3.28)

$$
1 - \frac{\omega_0^2 - n_1^2}{\omega_0^2 - n_2^2}\frac{n_2^2}{n_1^2}\delta = 0 \tag{3.30}
$$

Eliminating ω_0 from Eqs. (3.29) and (3.30) yields

$$\chi_1\chi_2 n_1^2 n_2^2(1-\delta)^2 - \delta\left\{(n_1^2 - n_2^2)^2 + \mu^2 n_1^2 n_2^2(1-\delta)[1-(n_2/n_1)^2\delta]\right\} = 0 \qquad (3.31)$$

Equation (3.31) gives the toe of the amplitude characteristics.*

(c) Stability Investigation

The stability of the steady-state solutions is tested as before. By a procedure analogous to that of the preceding section, we obtain the characteristic equation (3.20). After some algebraic manipulation, the coefficients of Eq. (3.20) are found as

$$\begin{aligned}
p &= -\mu\omega_0 n_1[2 - 2(n_2/n_1)^2\delta - r_{u0}^2] \\
q &= [2(1-\chi_1\chi_2)\omega_0^2 - (n_1^2 + n_2^2)^2]^2 + \mu^2\omega_0^2 n_1^2\left\{[1-(n_2/n_1)^2\delta - \tfrac{3}{4}r_{u0}^2]\right. \\
&\quad \left.\times[1-(n_2/n_1)^2\delta - \tfrac{1}{4}r_{u0}^2] + 2(1-\chi_1\chi_2)(n_2/n_1)^2\delta r_{u0}^2\right\} \qquad (3.32) \\
r &= \tfrac{1}{2}\mu\omega_0 n_1(\omega_0^2 - n_2^2)(1-\chi_1\chi_2)r_{u0}^2[2(1-\chi_1\chi_2)\omega_0^2 - (n_1^2 + n_2^2) \\
&\quad + \mu^2\omega_0^2 n_2^4(n_1^2 - n_2^2)\delta^2/n_1^2(\omega_0^2 - n_2^2)^2]
\end{aligned}$$

Then the stability conditions are given by the conditions (3.21). The stability conditions of the state that $r_{u0} = r_{v0} = 0$ are also obtained analogously (see Appendix III).

(d) Numerical Example

We consider a system in which

$$\mu = 0.1 \qquad k = 0.04 \qquad \text{and} \qquad (n_2/n_1)^2\delta = 0.5$$

---

* This equation coincides with that which gives the stability limit of the steady state u = v = 0 (see Appendix III).

Since k is small, the mutual indectance N is also small [see Eq. (2.22)]. Therefore, when $n_1 \cong n_2$, the difference between ω_1 and ω_2 becomes small and we may expect the internal resonance. Substituting these values of the parameters into Eq. (3.29) gives the relation between n_2/n_1 and ω_0/n_1. The result is shown in Fig. 3.5a. The entrained frequency ω_0/n_1 varies continuously as indicated in the figure (cf. Fig. 2.2). The amplitude characteristic is calculated by using Eqs. (3.28) and shown in Fig. 3.5b. To fix the values of $\mathcal{X}_1$ and $\mathcal{X}_2$, it is chosen that $L_1 = L_2$ (see the footnote of p.16). The amplitude r_{u0} dips when $n_1 \cong n_2$, and simultaneously an increase in the amplitude r_{v0} results.* The phase characteristic of the entrained oscillation is shown in Fig. 3.5c.

3.4 Concluding Remarks

Two typical cases of the internal resonance in a self-oscillatory system have been discussed. When the ratio between two natural frequencies is in the neighborhood of 1/3, the two natural frequency components are entrained mutually, i.e., internal resonance occurs. The entrained oscillation is characterized by the waveform which has the third-harmonic component predominantly.

When two natural frequencies are close each other, they are also entrained mutually. This type of internal resonance occurs when the two resonant circuits have nearly equal resonant frequencies and are weakly coupled. The entrained oscillation has a single harmonic component. The entrained frequency and amplitude characteristics vary continuously with varying frequencies of the resonant circuits. When the internal resonance occurs, the amplitude shows the dip phenomenon.

---

* This phenomenon is used in the absorption-type frequency meter.

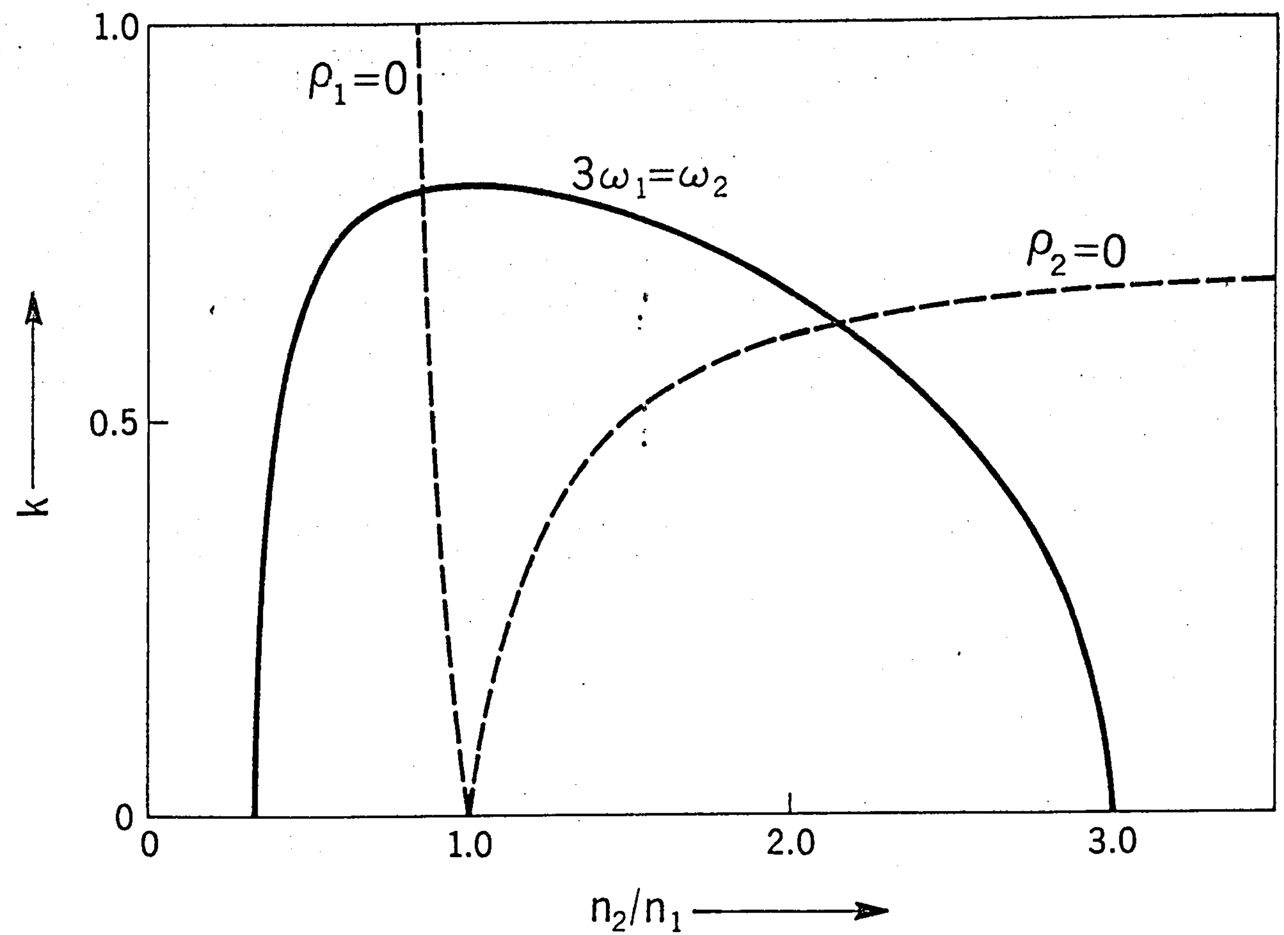

Fig. 3.1. The relationship between k and n_2/n_1 when $3\omega_1 = \omega_2$ is satisfied.

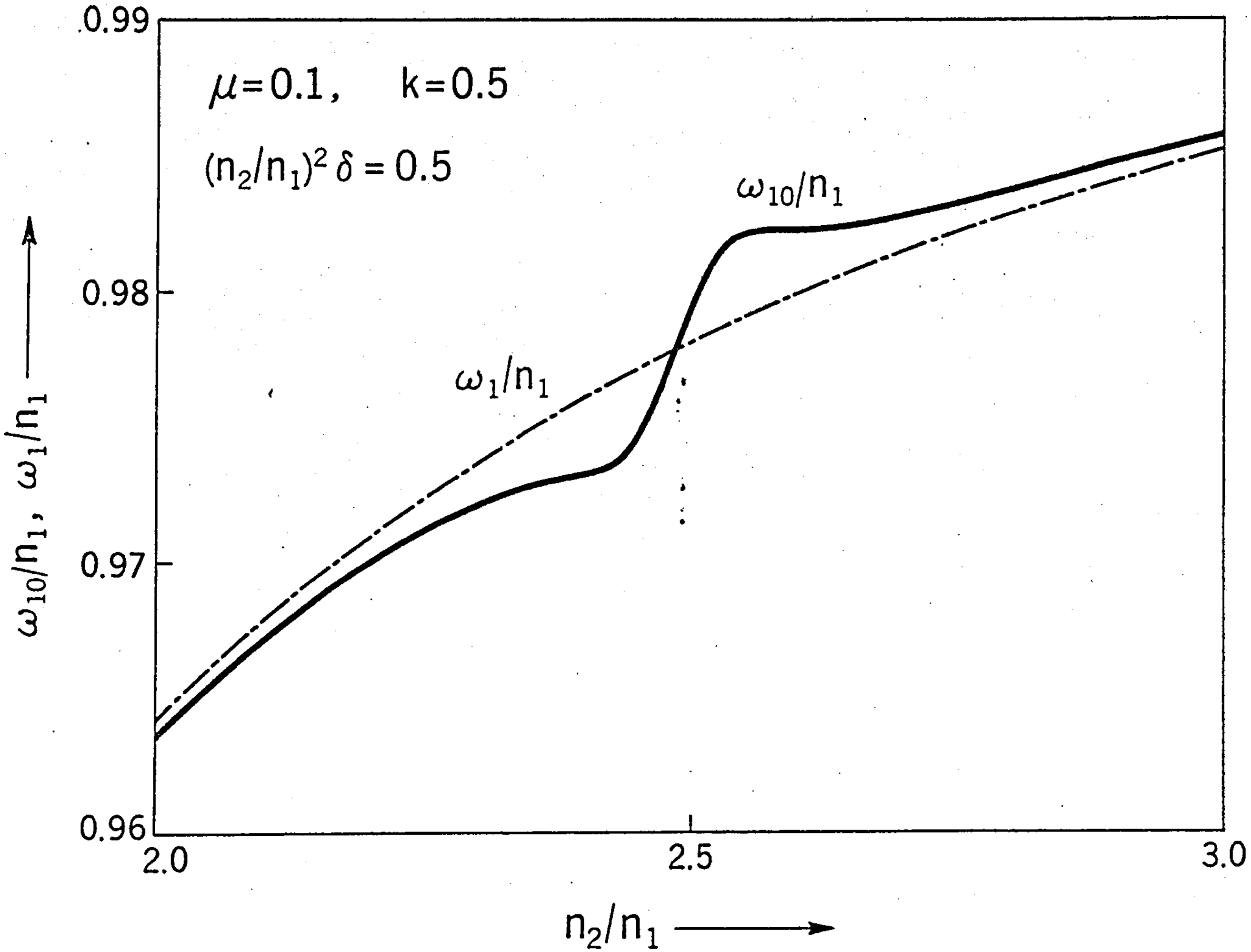

Fig. 3.2(a). Frequency characteristic of the entrained oscillation ($3\omega_1 \cong \omega_2$).

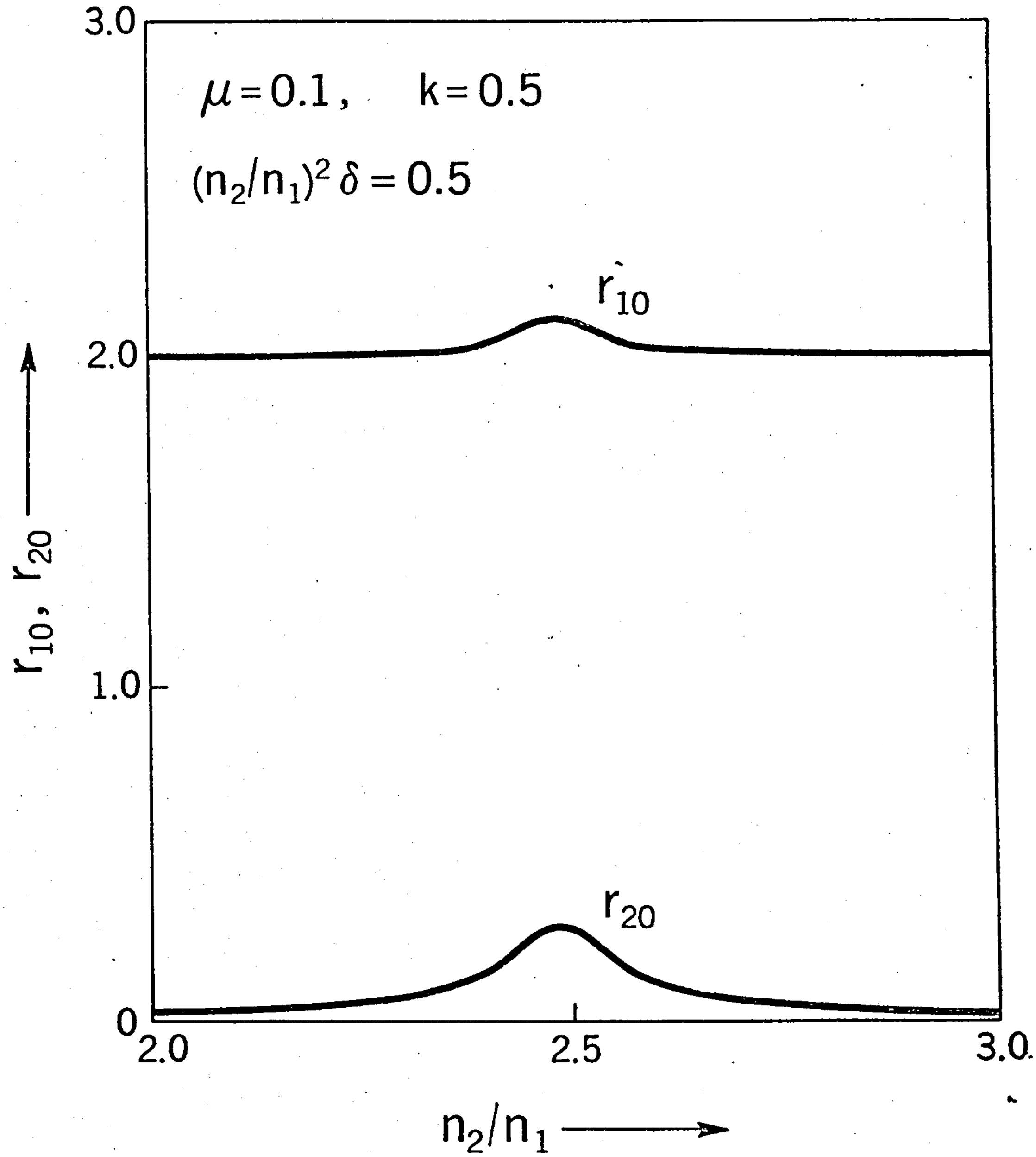

Fig. 3.2(b). Amplitude characteristic (r_{10} and r_{20}) of the entrained oscillation ($3\omega_1 \approxeq \omega_2$).

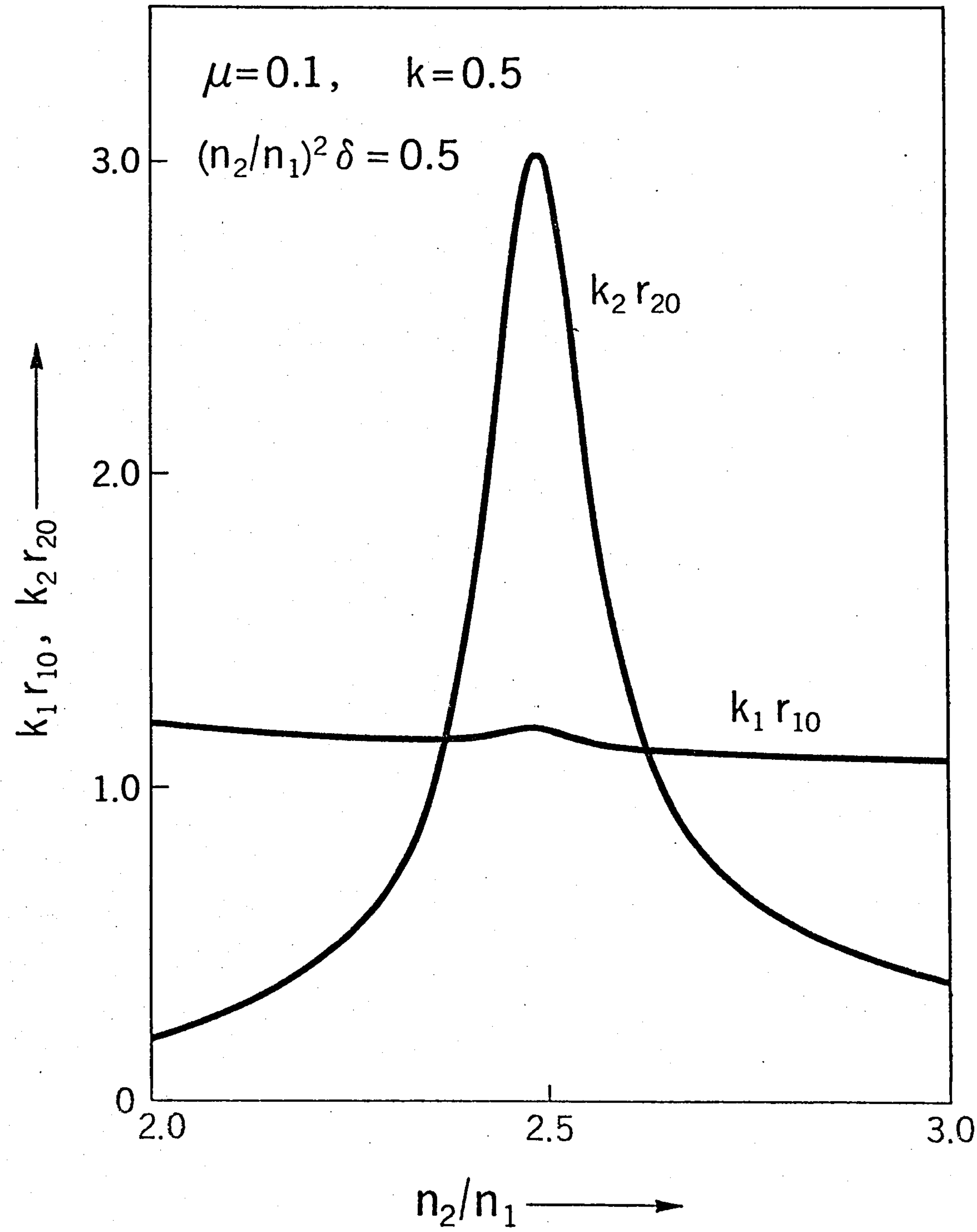

Fig. 3.2(c). Amplitude characteristic ($k_1 r_{10}$ and $k_2 r_{20}$) of the entrained oscillation ($3\omega_1 \cong \omega_2$).

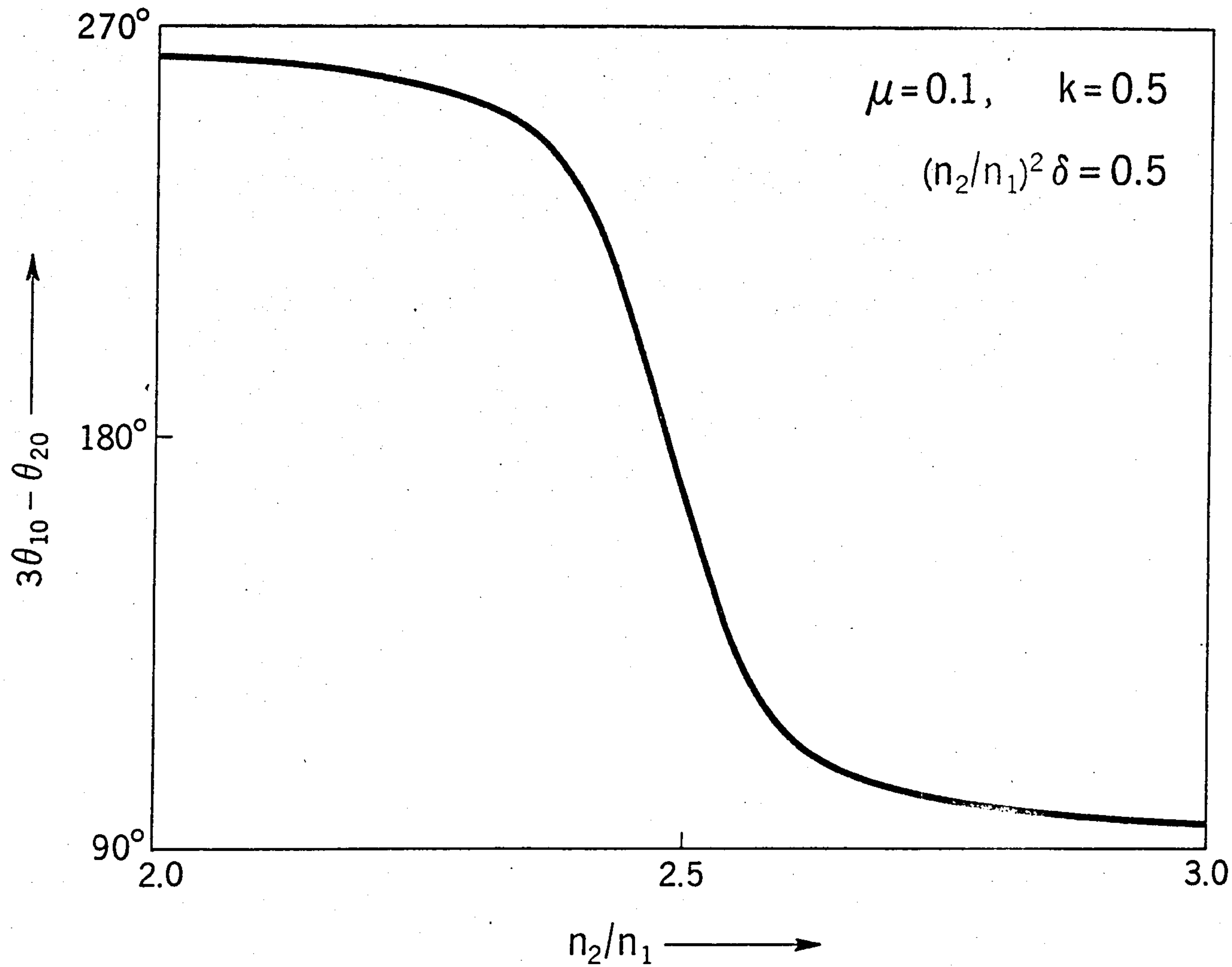

Fig. 3.2(d). Phase characteristic of the entrained oscillation ($3\omega_1 \cong \omega_2$).

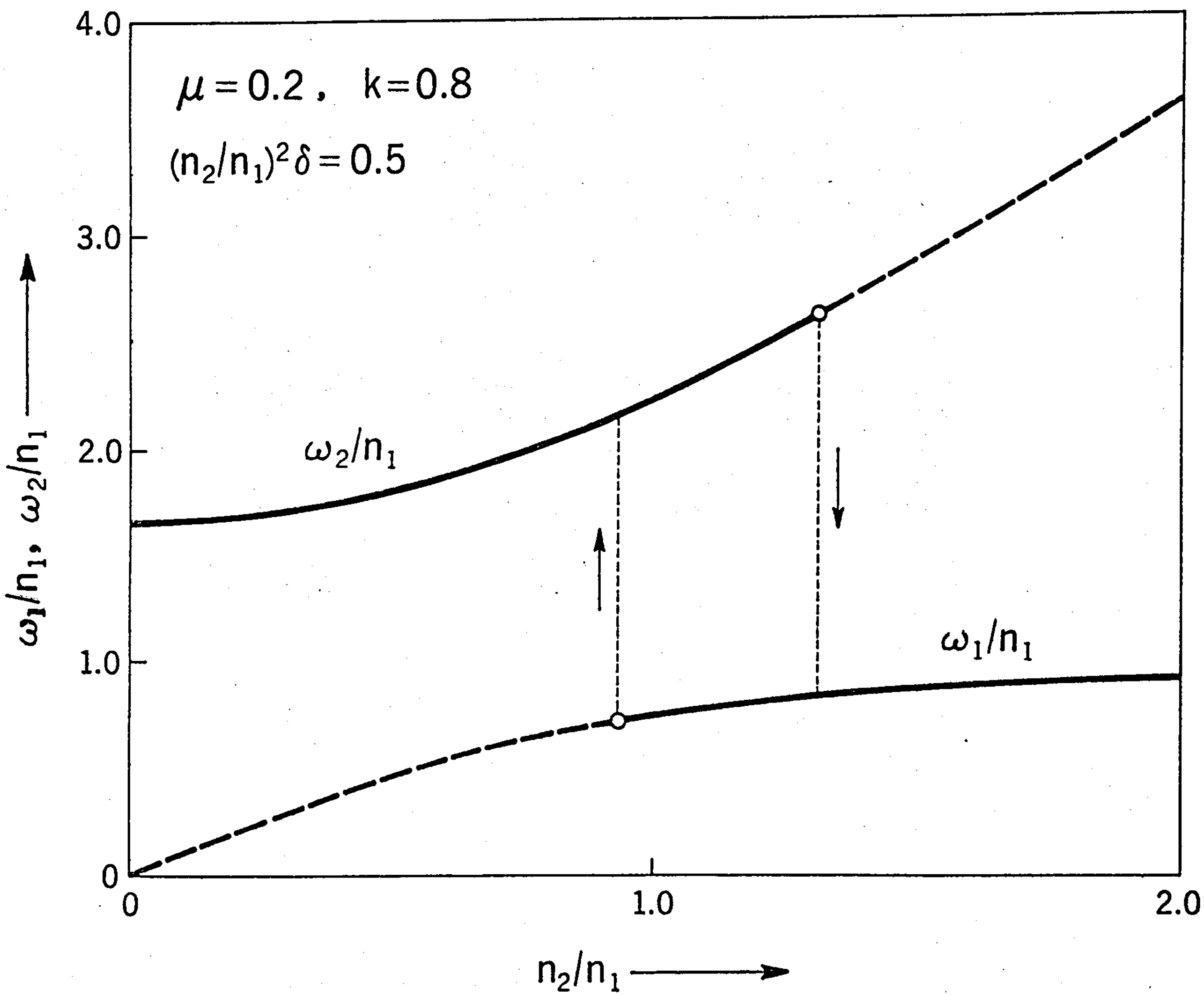

Fig. 3.3(a). Frequency characteristic of the entrained oscillation ($3\omega_1 \cong \omega_2$).

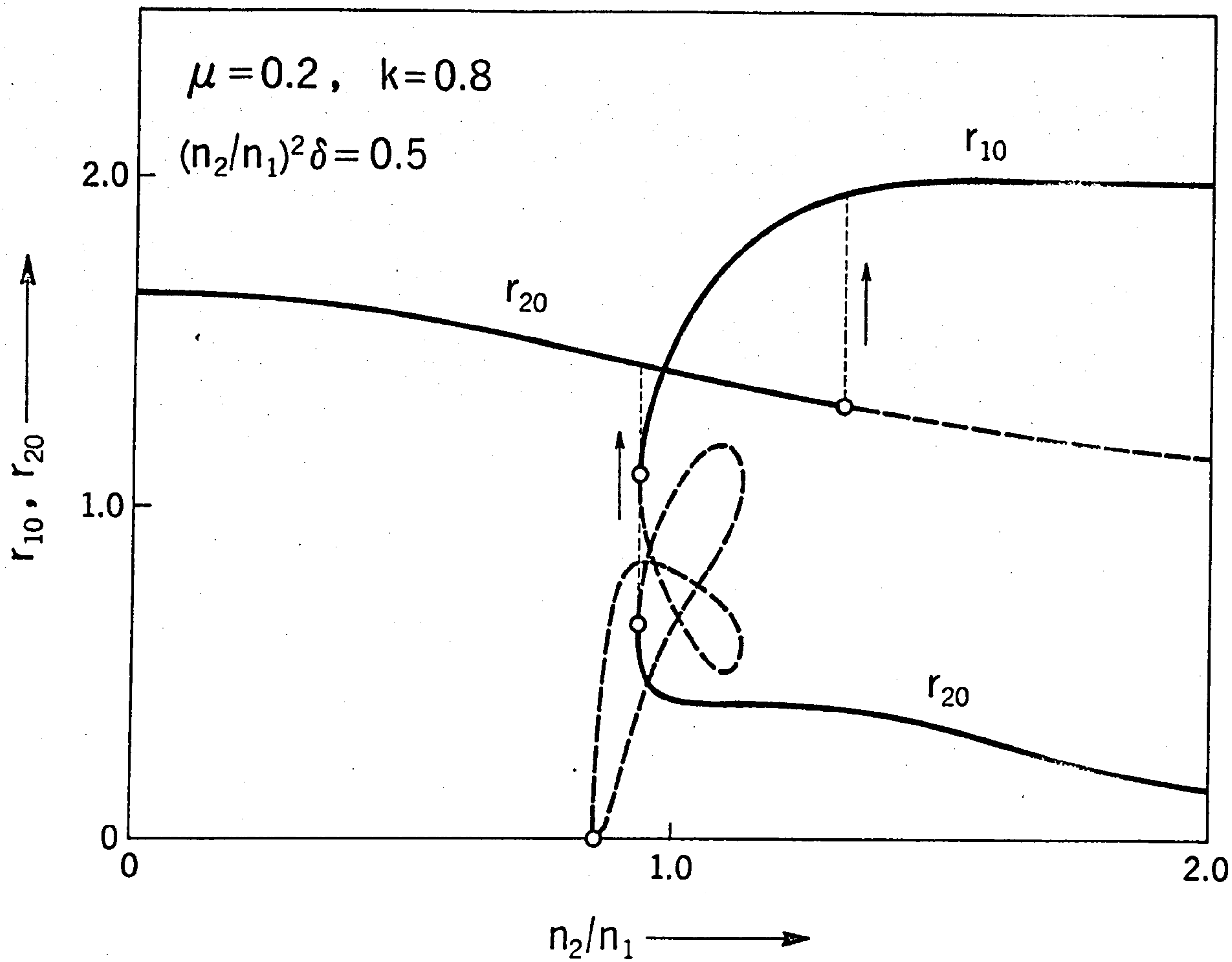

Fig. 3.3(b). Amplitude characteristic (r_{10} and r_{20}) of the entrained oscillation ($3\omega_1 \cong \omega_2$).

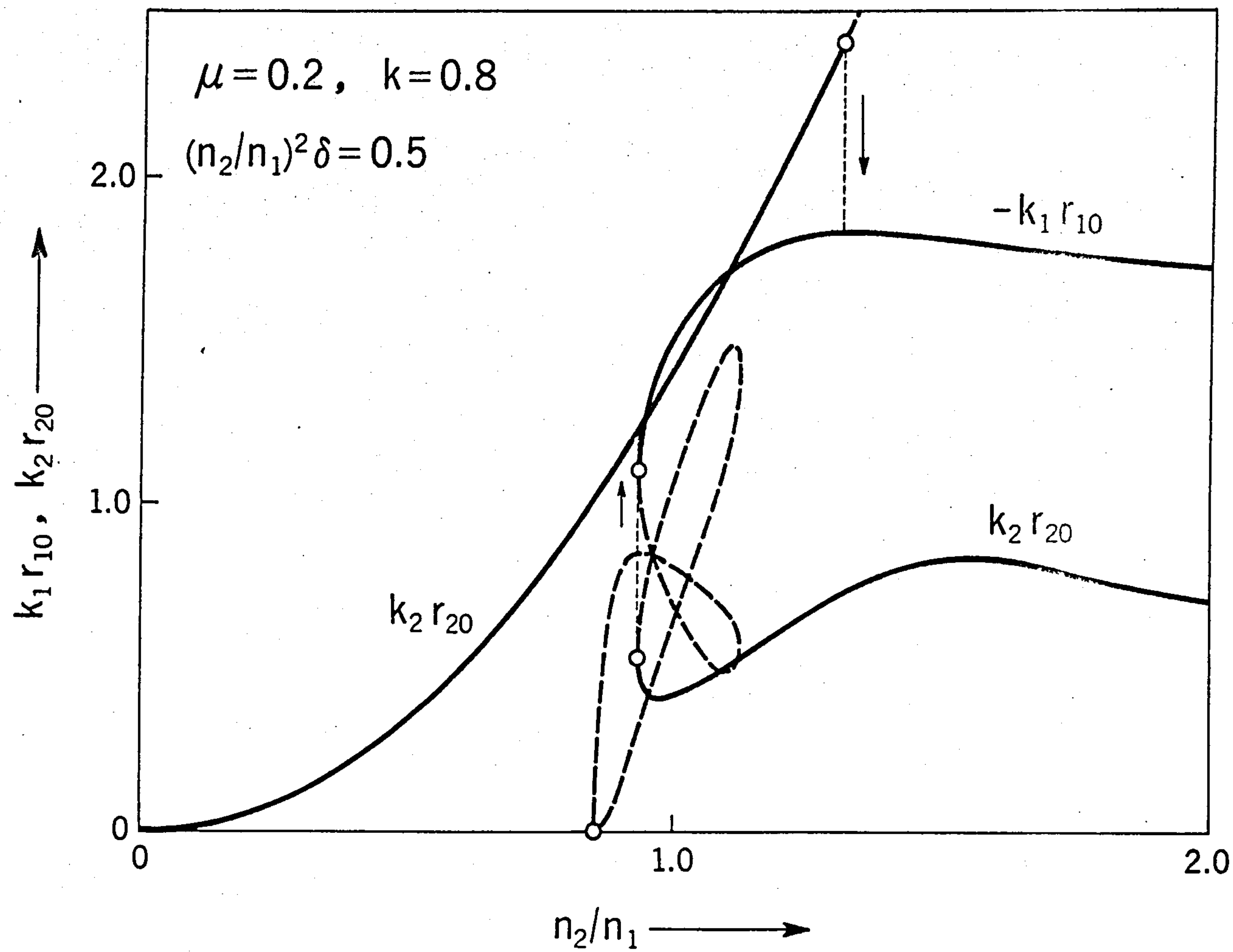

Fig. 3.3(c). Amplitude characteristic (k_1r_{10} and k_2r_{20}) of the entrained oscillation ($3\omega_1 \cong \omega_2$).

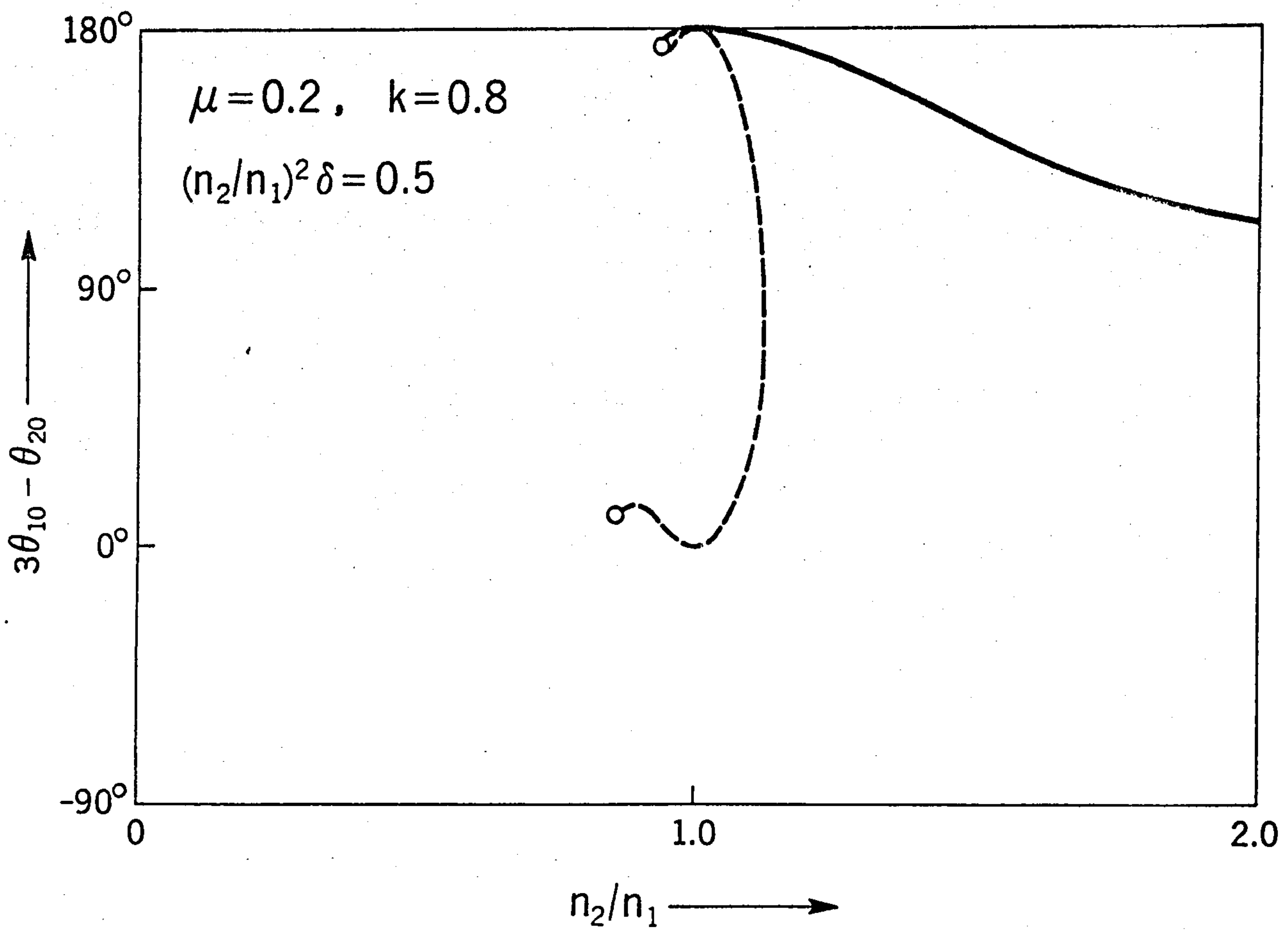

Fig. 3.3(d). Phase characteristic of the entrained oscillation ($3\omega_1 \approxeq \omega_2$).

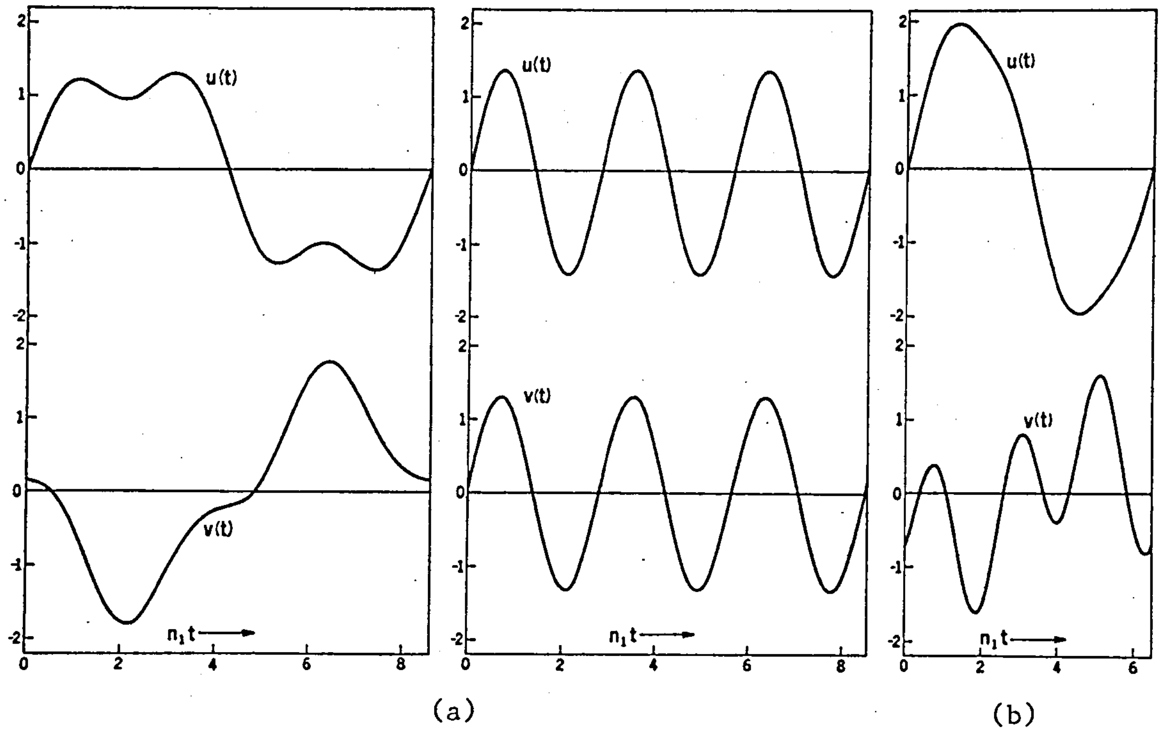

Fig. 3.4. Waveforms of the self-excited oscillations which occur when $3\omega_1 \cong \omega_2$.

(a) $\mu = 0.2$, $k = 0.8$, $(n_2/n_1)^2\delta = 1.0$, $n_2/n_1 = 1.0$

(b) $\mu = 0.1$, $k = 0.5$, $(n_2/n_1)^2\delta = 1.0$, $n_2/n_1 = 2.5$

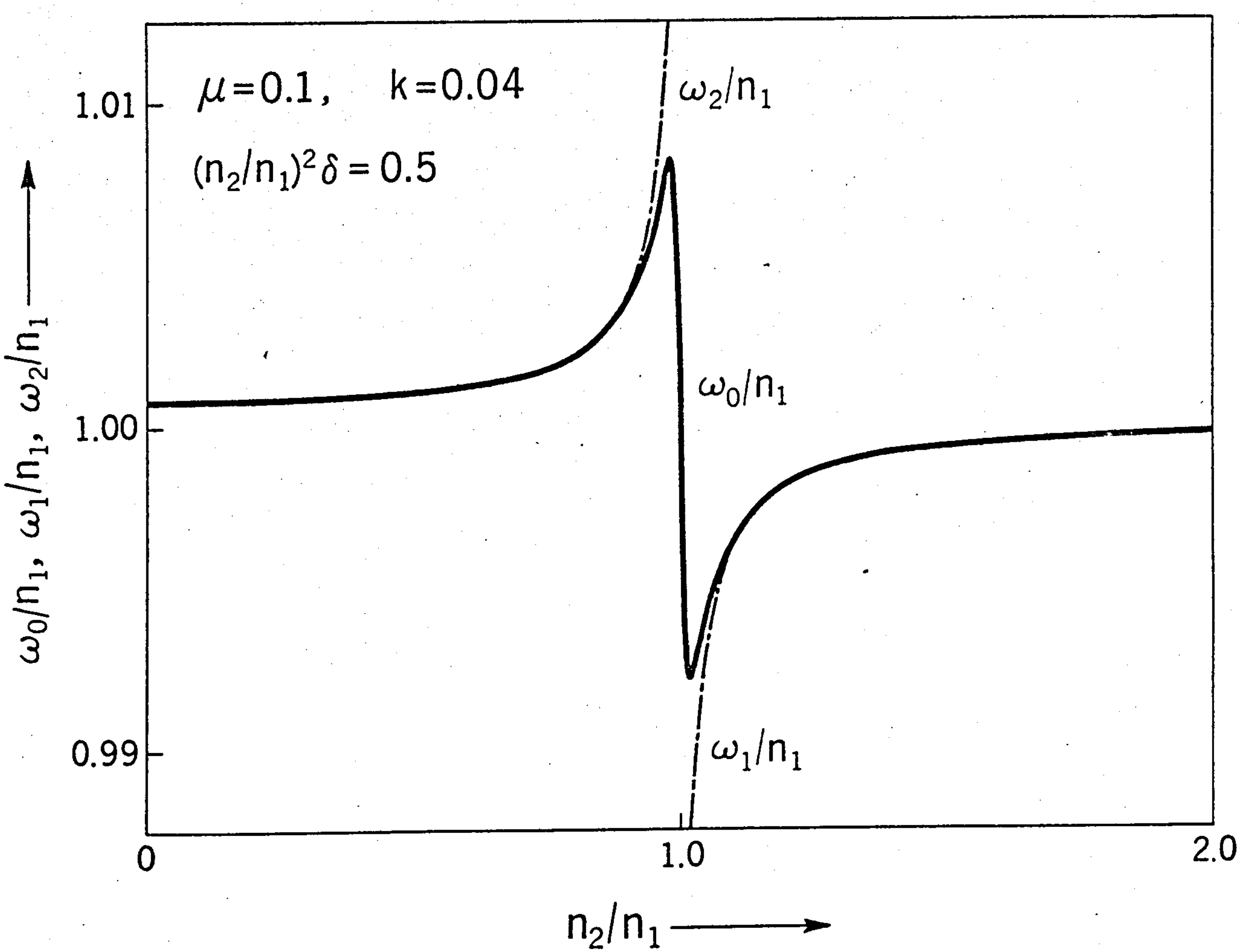

Fig. 3.5(a). Frequency characteristic of the entrained oscillation ($\omega_1 \cong \omega_2$).

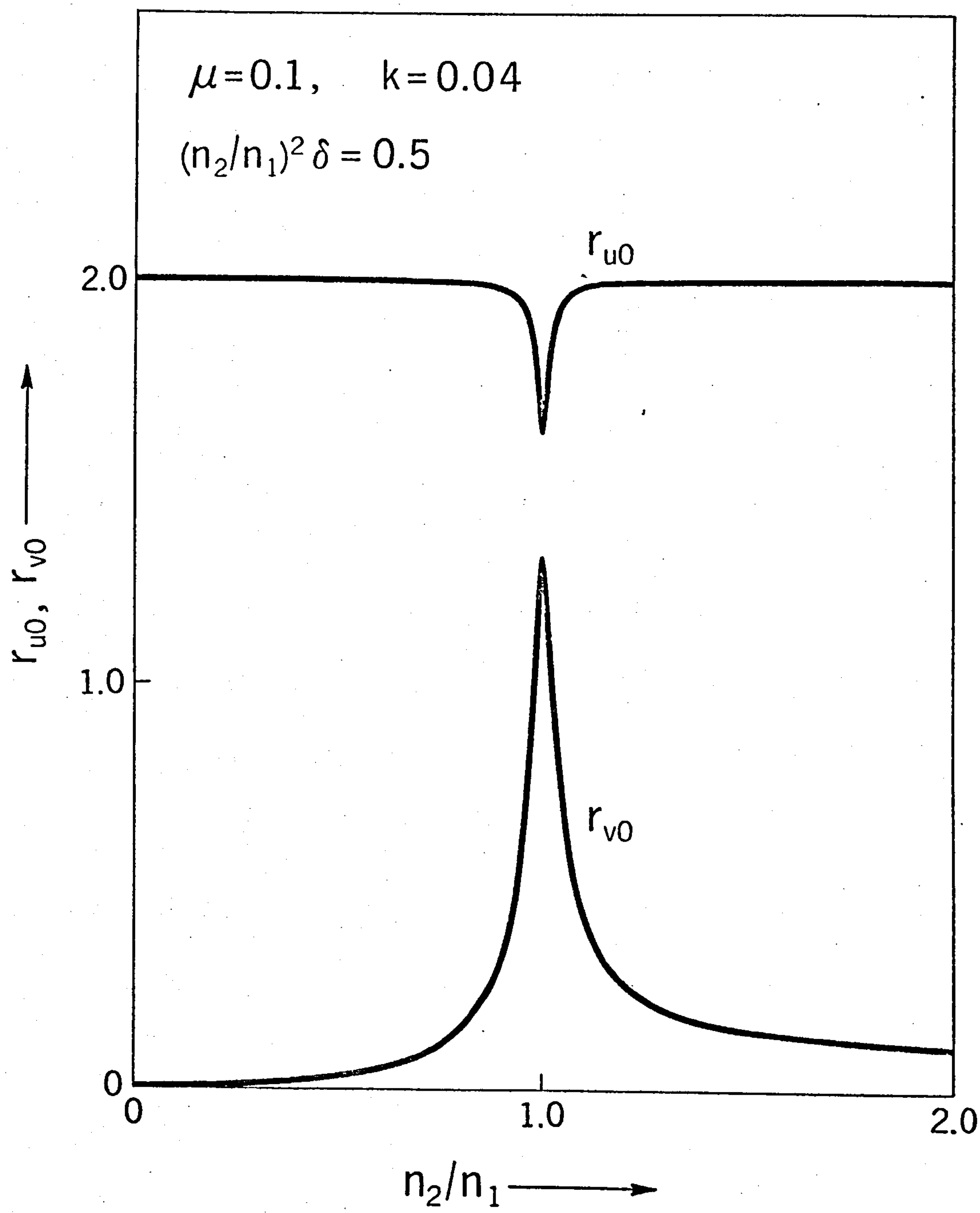

Fig. 3.5(b). Amplitude characteristic of the entrained oscillation ($\omega_1 \cong \omega_2$).

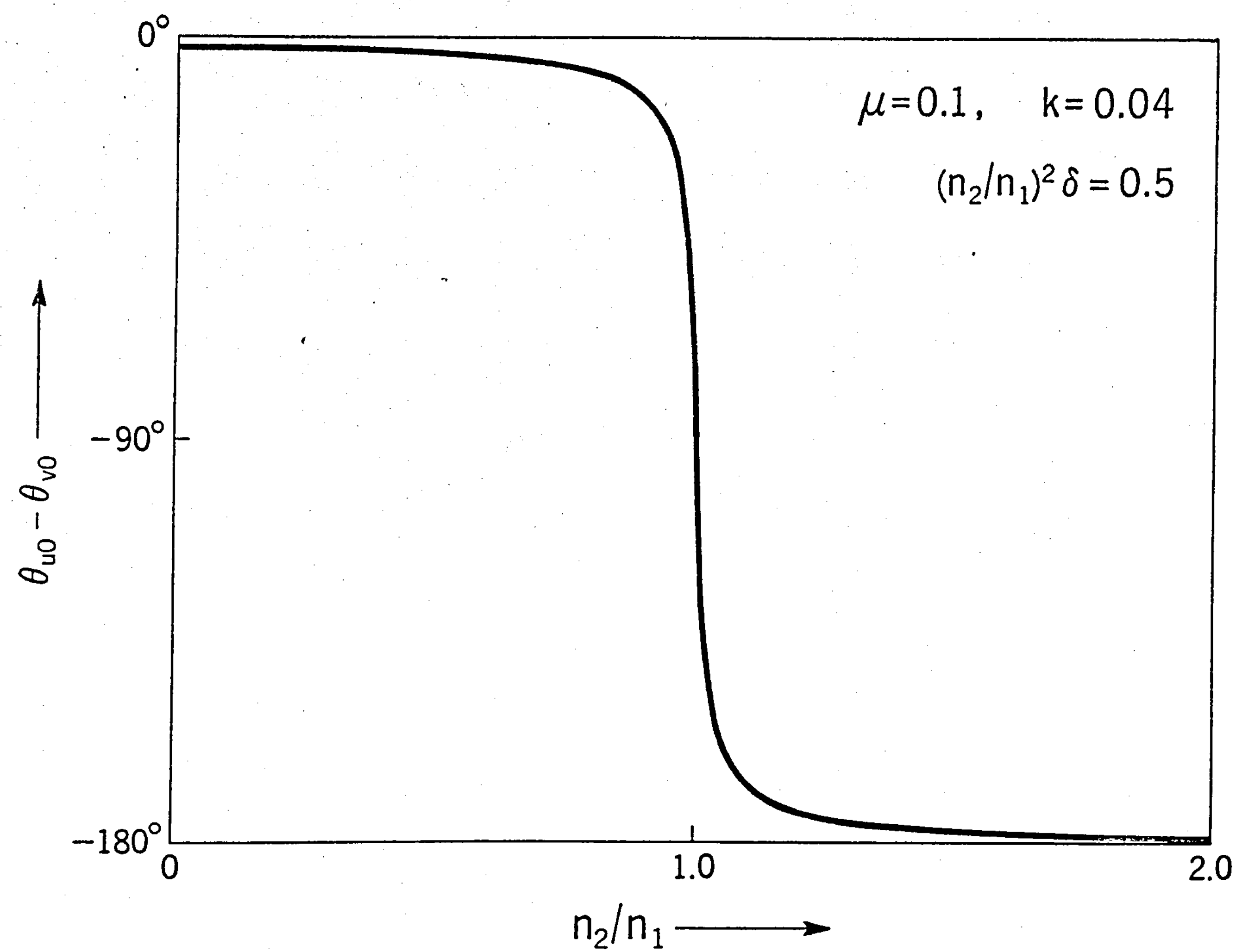

Fig. 3.5(c). Phase characteristic of the entrained oscillation ($\omega_1 \cong \omega_2$).

CHAPTER 4

FORCED OSCILLATIONS IN A SELF-OSCILLATORY SYSTEM WITHOUT INTERNAL RESONANCE

4.1 Introduction

In this chapter we study the behavior of a self-oscillatory system under the influence of an external sinusoidal force. We assume that no internal resonance occurs. Forced oscillations in a self-oscillatory system with one degree of freedom have been studied in a number of reports [6, 10, 26, 27, 35]. The natural frequency of the self-oscillatory system is entrained by the driving frequency provided that these two frequencies are not far different. If their difference is large enough, an almost periodic oscillation occurs. The natural frequency of the system is also entrained by a frequency which is an integral multiple or submultiple of the driving frequency when the ratio between the natural frequency and the driving frequency is in the neighborhood of an integer or a fraction. These phenomena of frequency entrainment, i.e., the external resonances are called the harmonic, higher-harmonic, and subharmonic entrainments, respectively [10].

The frequency entrainment also occurs in a system with two degrees of freedom. Since the system under consideration has two natural frequencies, one may expect the occurrence of the harmonic, higher-harmonic, and subharmonic entrainments between the driving frequency and one of the two natural frequencies. If a component having the other natural frequency also builds up, the resulting oscillation becomes almost periodic; on the other hand, if it doesn't build up, the resulting oscillation is periodic [5, 21]. If there exists a certain relationship among the driving frequency and the two natural frequencies [24, 36], for instance, if the driving frequency is in the neighborhood of the half of the sum of the two natural frequencies, this mean frequency is entrained by the driving frequency. The resulting oscillation containing three components of frequencies is generally almost periodic.

The averaging method is applied to the analysis of the standard form of equations (1.21) and (1.25). Depending on the types of the external resonance, different forms of the autonomous systems are derived. The stability is tested by using the Routh-Hurwitz criterion.

4.2 Derivation of Autonomous Systems by Using the Averaging Method

In this section we derive autonomous systems by using the averaging method. If the driving frequency ω is in the neighborhood of one of the two natural frequencies ω_1 and ω_2, one may expect that the natural frequency is entrained by the driving frequency ω. In this case the standard form of Eqs. (1.25) must be considered. If not so, the standard form of Eqs. (1.21) must be considered (see Sec. 1.3).

(a) Autonomous Systems Derived When $\omega \cong \omega_i$

To investigate the frequency entrainment, we make use of the expansion

$$\begin{aligned} \omega_1 &= \omega_{10} + \mu\omega_{11} + \cdots \\ \omega_2 &= \omega_{20} + \mu\omega_{21} + \cdots \end{aligned} \tag{4.1}$$

The frequencies ω_{10} and ω_{20} are in the neighborhood of ω_1 and ω_2, respectively, and one of them is equal to .* We assume that the entrained oscillation consists

---

* When $\omega \cong \omega_1$, for instance, we set $\omega_{10} = \omega$. Then it is shown that ω_{20} is equal to ω_2 and $\omega_{21}, \omega_{22}, \cdots$ vanish (see Sec. 4.3.1).

of two components of frequencies ω_{10} and ω_{20} [see Eqs. (4.7)]. Substituting Eqs. (4.1) into Eqs. (1.25) gives

$$\begin{aligned} \ddot{x} + \omega_{10}^2 x &= \mu F_{30}(x, y, \dot{x}, \dot{y}, t) + O(\mu^2) \\ \ddot{y} + \omega_{20}^2 y &= \mu G_{30}(x, y, \dot{x}, \dot{y}, t) + O(\mu^2) \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F_{30}(x, y, \dot{x}, \dot{y}, t) &= F_3(x, y, \dot{x}, \dot{y}, t) - 2\omega_{10}\omega_{11}x \\ &= \frac{\omega_1^2}{n_1}\frac{1}{k_2 - k_1}\Big\{k_2[1 - (x + y)^2](\dot{x} + \dot{y}) \\ &\quad + \delta(k_1\dot{x} + k_2\dot{y}) + k_2 n_1 B_1 \cos \omega t\Big\} - 2\omega_{10}\omega_{11}x \\ G_{30}(x, y, \dot{x}, \dot{y}, t) &= G_3(x, y, \dot{x}, \dot{y}, t) - 2\omega_{20}\omega_{21}y \\ &= \frac{\omega_2^2}{n_1}\frac{1}{k_1 - k_2}\Big\{k_1[1 - (x + y)^2](\dot{x} + \dot{y}) \\ &\quad + \delta(k_1\dot{x} + k_2\dot{y}) + k_1 n_1 B_1 \cos \omega t\Big\} - 2\omega_{20}\omega_{21}y \end{aligned} \tag{4.3}$$

The generating system obtained by putting $\mu = 0$ in Eqs. (4.2) is

$$\begin{aligned} \ddot{x} + \omega_{10}^2 x &= 0 \\ \ddot{y} + \omega_{20}^2 y &= 0 \end{aligned} \tag{4.4}$$

The general solution of Eqs. (4.4) is

$$\begin{aligned} x(t) &= r_1 \cos (\omega_{10}t + \theta_1) \\ y(t) &= r_2 \cos (\omega_{20}t + \theta_2) \end{aligned} \tag{4.5}$$

where r_1, r_2, θ_1, and θ_2 are integration constants.

When $\mu \neq 0$, we write for x and y in Eqs. (4.2) as

$$x(t) = r_1(t) \cos [\omega_{10}t + \theta_1(t)]$$

$$
\begin{aligned}
y(t) &= r_2(t) \cos [\omega_{20}t + \theta_2(t)] \\
\dot{x}(t) &= -\omega_{10} r_1(t) \sin [\omega_{10}t + \theta_1(t)] \\
\dot{y}(t) &= -\omega_{20} r_2(t) \sin [\omega_{20}t + \theta_2(t)]
\end{aligned} \tag{4.6}
$$

We assume that, for a small value of μ, both the amplitudes $r_1(t)$, $r_2(t)$, and the phase angles $\theta_1(t)$, $\theta_2(t)$ are slowly varying functions of t. By using Eqs. (1.14) and (4.6), the solution of Eqs. (1.24) is written as

$$
\begin{aligned}
u(t) &= r_1(t) \cos [\omega_{10}t + \theta_1(t)] + r_2(t) \cos [\omega_{20}t + \theta_2(t)] \\
v(t) &= k_1 r_1(t) \cos [\omega_{10}t + \theta_1(t)] + k_2 r_2(t) \cos [\omega_{20}t + \theta_2(t)]
\end{aligned} \tag{4.7}
$$

Substituting Eqs. (4.6) into Eqs. (4.2) gives

$$
\begin{aligned}
\dot{r}_1 \sin (\omega_{10}t + \theta_1) + r_1\dot{\theta}_1 \cos (\omega_{10}t + \theta_1) &= -\frac{\mu}{\omega_{10}} f_{30}(r_1, r_2, \theta_1, \theta_2, t) \\
\dot{r}_2 \sin (\omega_{20}t + \theta_2) + r_2\dot{\theta}_2 \cos (\omega_{20}t + \theta_2) &= -\frac{\mu}{\omega_{20}} g_{30}(r_1, r_2, \theta_1, \theta_2, t)
\end{aligned} \tag{4.8}
$$

where $f_{30}(r_1, r_2, \theta_1, \theta_2, t)$ and $g_{30}(r_1, r_2, \theta_1, \theta_2, t)$ are the functions obtained by substitution of Eqs. (4.6) into F_{30} and G_{30} of Eqs. (4.3), respectively. From Eqs. (4.6) we obtain

$$
\begin{aligned}
\dot{r}_1 \cos (\omega_{10}t + \theta_1) - r_1\dot{\theta}_1 \sin (\omega_{10}t + \theta_1) &= 0 \\
\dot{r}_2 \cos (\omega_{20}t + \theta_2) - r_2\dot{\theta}_2 \sin (\omega_{20}t + \theta_2) &= 0
\end{aligned} \tag{4.9}
$$

From Eqs. (4.8) and (4.9) we obtain the simultaneous differential equations regarding r_1, r_2, θ_1, and θ_2 as :

$$
\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_{10}} f_{30}(r_1, r_2, \theta_1, \theta_2, t) \sin (\omega_{10}t + \theta_1) \\
\dot{r}_2 &= -\frac{\mu}{\omega_{20}} g_{30}(r_1, r_2, \theta_1, \theta_2, t) \sin (\omega_{20}t + \theta_2)
\end{aligned}
$$

$$r_1\dot{\theta}_1 = -\frac{\mu}{\omega_{10}} f_{30}(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{10}t + \theta_1) \tag{4.10}$$

$$r_2\dot{\theta}_2 = -\frac{\mu}{\omega_{20}} g_{30}(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{20}t + \theta_2)$$

Equations (4.10) show that r_i and θ_i $(i = 1, 2)$ are both of the order of μ; therefore, as expected, r_i and θ_i are slowly varying functions of t. Hence, upon application of the averaging method, Eqs. (4.10) can be transformed into autonomous differential equations, i.e.,

$$\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_{10}} \lim_{T\to\infty}\frac{1}{T}\int_0^T f_{30}(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_{10}t + \theta_1)dt \\
\dot{r}_2 &= -\frac{\mu}{\omega_{20}} \lim_{T\to\infty}\frac{1}{T}\int_0^T g_{30}(r_1, r_2, \theta_1, \theta_2, t)\sin(\omega_{20}t + \theta_2)dt \\
r_1\dot{\theta}_1 &= -\frac{\mu}{\omega_{10}} \lim_{T\to\infty}\frac{1}{T}\int_0^T f_{30}(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_{10}t + \theta_1)dt \\
r_2\dot{\theta}_2 &= -\frac{\mu}{\omega_{20}} \lim_{T\to\infty}\frac{1}{T}\int_0^T g_{30}(r_1, r_2, \theta_1, \theta_2, t)\cos(\omega_{20}t + \theta_2)dt
\end{aligned} \tag{4.11}$$

The integration is to be performed with respect to the explicitly appearing t in the integrant.

A1:* When $\omega = \omega_{10}$, performing the integration of Eqs. (4.11) yields†

$$\dot{r}_1 = \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - r_1^2 - 2r_2^2)r_1 - \frac{4}{\omega}n_1B_1\sin\theta_1] \equiv h_1(r_1, r_2, \theta_1, \theta_2)$$

$$\dot{r}_2 = \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2r_1^2 - r_2^2)r_2 \equiv h_2(r_1, r_2, \theta_1, \theta_2)$$

---

* The symbol corresponds to the classification of the type in Tables 4.1 to 4.4.

† We see from Eqs. (4.1) that $\omega_1 - \omega = \omega_1 - \omega_{10} = \mu\omega_{11} + O(\mu^2)$

$$r_1\dot{\theta}_1 = \mu(\omega_{11}r_1 - \frac{\omega_1^2}{2\omega}\frac{k_2}{k_2 - k_1}B_1\cos\theta_1) \equiv r_1h_3(r_1, r_2, \theta_1, \theta_2) \tag{4.12}$$

$$r_2\dot{\theta}_2 = \mu\omega_{21}r_2 \equiv r_2h_4(r_1, r_2, \theta_1, \theta_2)$$

where ρ_1 and ρ_2 are given by Eqs. (2.8).

A2: When $\omega = \omega_{20}$, we obtain

$$\begin{aligned}
\dot{r}_1 &= \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - r_1^2 - 2r_2^2)r_1 \\
\dot{r}_2 &= \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\rho_2 - 2r_1^2 - r_2^2)r_2 - \frac{4}{\omega}n_1B_1\sin\theta_2] \\
r_1\dot{\theta}_1 &= \mu\omega_{11}r_1 \\
r_2\dot{\theta}_2 &= \mu[\omega_{21}r_2 - \frac{\omega_2^2}{2\omega}\frac{k_1}{k_1 - k_2}B_1\cos\theta_2]
\end{aligned} \tag{4.13}$$

The averaged equations (4.12) and (4.13) are derived under the assumption that no internal resonance occurs. Therefore, ω_{20} is neither equal to ω_{10} nor $3\omega_{10}$. The internal resonance will be discussed in Chaps. 5 and 6.

(b) Autonomous Systems Derived When ω Is Not near ω_i

We consider the standard form of Eqs. (1.21). When the ratio between the driving frequency and one of the two natural frequencies ω_1 and ω_2 is in the neighborhood of an integer (different from unity) or a fraction, one may expect the occurrence of higher-harmonic or subharmonic entrainment. To investigate this phenomenon, we assume that the entrained oscillation consists of a component of the driving frequency ω and two components of frequencies ω_{10} and ω_{20}, which are slightly different from ω_1 and ω_2 [see Eqs. (4.16)].* We develop ω_1 and ω_2

---

* The frequency ω_{i0} (i = 1, 2) may be equal to ω_i (see Sec. 4.3).

in power series of μ, as shown in Eqs. (4.1). Substituting Eqs. (4.1) into Eqs. (1.21) gives

$$
\begin{aligned}
&\ddot{x} + \omega_{10}^2 x = \mu F_{20}(x, y, \dot{x}, \dot{y}, t) \\
&\ddot{y} + \omega_{20}^2 y = \mu G_{20}(x, y, \dot{x}, \dot{y}, t)
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
F_{20}(x, y, \dot{x}, \dot{y}, t) &= F_2(x, y, \dot{x}, \dot{y}, t) - 2\omega_{10}\omega_{11}x \\
&= \frac{\omega_1^2}{n_1}\frac{1}{k_2 - k_1}\Big\{k_2[1 - (x + y + A_1 \cos \omega t)^2](\dot{x} + \dot{y} - \omega A_1 \sin \omega t) \\
&\quad + \delta(k_1\dot{x} + k_2\dot{y} - \omega A_2 \sin \omega t)\Big\} - 2\omega_{10}\omega_{11}x \\
G_{20}(x, y, \dot{x}, \dot{y}, t) &= G_2(x, y, \dot{x}, \dot{y}, t) - 2\omega_{20}\omega_{21}y \\
&= \frac{\omega_2^2}{n_1}\frac{1}{k_1 - k_2}\Big\{k_1[1 - (x + y + A_1 \cos \omega t)^2](\dot{x} + \dot{y} - \omega A_1 \sin \omega t) \\
&\quad + \delta(k_1\dot{x} + k_2\dot{y} - \omega A_2 \sin \omega t)\Big\} - 2\omega_{20}\omega_{21}y
\end{aligned}
\tag{4.15}
$$

The generating solution of Eqs. (4.14) takes the same form as Eqs. (4.5). Therefore, when $\mu \neq 0$, we assume that the solution of Eqs. (4.14) also takes the same form as Eqs. (4.6). From Eqs. (1.20) and (4.6), the solution of Eqs. (1.7) is written as *

$$
\begin{aligned}
u(t) &= r_1(t) \cos [\omega_{10}t + \theta_1(t)] + r_2(t) \cos [\omega_{20}t + \theta_2(t)] + A_1 \cos \omega t \\
v(t) &= k_1 r_1(t) \cos [\omega_{10}t + \theta_1(t)] + k_2 r_2(t) \cos [\omega_{20}t + \theta_2(t)] + A_2 \cos \omega t
\end{aligned}
\tag{4.16}
$$

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* As one sees from Eqs. (1.19), the amplitude A_1 of the harmonic oscillation becomes zero when $\omega = n_2$. Therefore, when ω is in the neighborhood of n_2, the term containing $\sin \omega t$ must be considered.

Substituting Eqs. (4.6) into Eqs. (4.14) and using the averaging method as we have done in Eqs. (4.8) through Eqs. (4.11), we obtain an autonomous system of the same form as Eqs. (4.11); i.e.,

$$\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_{10}} \lim_{T\to\infty} \frac{1}{T} \int_0^T f_{20}(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_{10}t + \theta_1)dt \\
\dot{r}_2 &= -\frac{\mu}{\omega_{20}} \lim_{T\to\infty} \frac{1}{T} \int_0^T g_{20}(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_{20}t + \theta_2)dt \\
r_1\dot{\theta}_1 &= -\frac{\mu}{\omega_{10}} \lim_{T\to\infty} \frac{1}{T} \int_0^T f_{20}(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{10}t + \theta_1)dt \\
r_2\dot{\theta}_2 &= -\frac{\mu}{\omega_{20}} \lim_{T\to\infty} \frac{1}{T} \int_0^T g_{20}(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{20}t + \theta_2)dt
\end{aligned} \tag{4.17}$$

where $f_{20}(r_1, r_2, \theta_1, \theta_2, t)$ and $g_{20}(r_1, r_2, \theta_1, \theta_2, t)$ are the functions obtained by substitution of Eqs. (4.6) into F_{20} and G_{20} of Eqs. (4.15), respectively. These functions f_{20} and g_{20} contain the terms having the following frequencies.*

$$\begin{aligned}
&\omega_{10},\ \omega_{20},\ 3\omega_{10},\ 3\omega_{20},\ 2\omega_{20} \pm \omega_{10},\ 2\omega_{10} \pm \omega_{20},\ \omega,\ 3\omega, \\
&2\omega_{10} \pm \omega,\ 2\omega_{20} \pm \omega,\ \omega_{10} \pm 2\omega,\ \omega_{20} \pm 2\omega,\ \omega_{10} \pm \omega_{20} \pm \omega
\end{aligned} \tag{4.18}$$

When a certain relationship exists among ω_{10}, ω_{20}, and ω, some of the frequencies listed in (4.18) may coincide with ω_{10} and ω_{20}. For instance, if $\omega = (\omega_{10} + \omega_{20})/2$, the frequency components $2\omega - \omega_{20}$ and $2\omega - \omega_{10}$ coincide with ω_{10} and ω_{20}, respectively. Therefore these terms do not vanish upon integration of Eqs. (4.17). In this case, one may expect a kind of resonance, i.e., the entrainment between ω and $(\omega_1 + \omega_2)/2$. These relationships including $\omega = \omega_{10}$ are summarized in Tables 4.1 to 4.4. The relationships listed in Table 4.1 are the cases where no internal resonance occurs, i.e., the ratio between ω_{10} and

---

* See the expanded form of f_{20} and g_{20} in Appendix I.

ω_{20} is not an integer or a fraction. Table 4.2 shows the cases where the internal resonance $3\omega_{10} = \omega_{20}$ occurs. The relationships in Table 4.2 are realized when the driving frequency is applied to the self-oscillatory system discussed in Sec. 3.2. Table 4.3 shows the internal resonance $s\omega_{10} = \omega_{20}$, where s is an integer (different from 1 and 3) or a fraction. If $\omega_{10} = \omega_{20}$, the derivation of the standard form of Eqs. (1.21) and (1.25) ceases to be meaningful (see Sec. 1.3). Therefore, the averaged equations (4.17) are of no use. Performing the integration of Eqs. (4.17), however, suggests us that the external resonances as listed in Table 4.4 occur.

In this chapter we treat the cases listed in Table 4.1. The internal and external resonances listed in Tables 4.2 and 4.3 will be discussed in Chap. 5. The internal and external resonances listed in Table 4.4 will be discussed in Chap. 6. Figure 4.1 shows the relationships among ω_{10}, ω_{20}, and ω listed in Tables 4.1 to 4.4. The coordinates in the figure are the ratios of the frequencies, ω_{10}/ω and ω_{20}/ω.* The symbols in the figure correspond to those in Tables 4.1 to 4.4.†

When no internal resonance occurs, performing the integration of Eqs. (4.17) gives the following averaged equations.

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* Since $\omega_{10} \leqq \omega_{20}$, we consider only the region $\omega_{10}/\omega \leqq \omega_{20}/\omega$ on the coordinates plane (see the footnote of p. 4).

† For k = 0.5 and $n_2/n_1 = 1.0$, we obtain $\omega_1/\omega_2 = 0.577$ (see Fig. 2.1). Then we obtain the following relation between ω_1/ω and ω_2/ω, i.e.,

$$\omega_1/\omega \equiv (\omega_1/\omega_2)(\omega_2/\omega) = 0.577\omega_2/\omega$$

This linear relation is plotted on the figure by the dotted line (see Table 4.5).

Table 4.1 Classification of the external resonance (without internal resonance)

Type	Relation among ω_{10}, ω_{20} and ω	Frequencies which are equal to ω_{10}	Frequencies which are equal to ω_{20}
A 1	$\omega = \omega_{10}$	———	———
A 2	$\omega = \omega_{20}$	———	———
A 3	$\omega = \omega_{10}/3$	3ω	———
A 4	$\omega = \omega_{20}/3$	———	3ω
A 5	$\omega = 3\omega_{10}$	$\omega - 2\omega_{10}$	———
A 6	$\omega = 3\omega_{20}$	———	$\omega - 2\omega_{20}$
A 7	$\omega = (\omega_{10} + \omega_{20})/2$	$2\omega - \omega_{20}$	$2\omega - \omega_{10}$
A 8	$\omega = (\omega_{20} - \omega_{10})/2$	$\omega_{20} - 2\omega$	$\omega_{10} + 2\omega$
A 9	$\omega = \omega_{10} + 2\omega_{20}$	$\omega - 2\omega_{20}$	$\omega - \omega_{10} - \omega_{20}$
A10	$\omega = 2\omega_{20} - \omega_{10}$	$2\omega_{20} - \omega$	$\omega_{10} - \omega_{20} + \omega$
A11	$\omega = 2\omega_{10} + \omega_{20}$	$\omega - \omega_{10} - \omega_{20}$	$\omega - 2\omega_{10}$
A12	$\omega = \pm\ (2\omega_{10} - \omega_{20})$	$\omega_{20} - \omega_{10} + \omega$	$2\omega_{10} + \omega$
A13	Non-Resonant Case	———	———

Table 4.2 Classification of the external resonance (internal resonance : $3\omega_{10} = \omega_{20}$)

Type	Relations among ω_{10}, ω_{20}, and ω	Frequencies which are equal to ω_{10}	Frequencies which are equal to ω_{20}
B1	$\omega = \omega_{10} = \frac{\omega_{20}}{3}$	$\omega_{20} - 2\omega_{10}$, ω	$3\omega_{10}$
B2	$\omega = 3\omega_{10} = \omega_{20}$	$\omega_{20} - 2\omega_{10}$	$3\omega_{10}$, ω
B3	$\omega = \frac{\omega_{10}}{3} = \frac{\omega_{20}}{9}$	$\omega_{20} - 2\omega_{10}$, 3ω	$3\omega_{10}$
B4	$\omega = 2\omega_{10} = \frac{2}{3}\omega_{20}$	$\omega_{20} - 2\omega_{10}$, $2\omega - \omega_{20}$	$3\omega_{10}$, $2\omega - \omega_{10}$
B5	$\omega = 5\omega_{10} = \frac{5}{3}\omega_{20}$	$\omega_{20} - 2\omega_{10}$, $2\omega_{10} - \omega$, $\omega - \omega_{10} - \omega_{20}$	$3\omega_{10}$, $\omega - 2\omega_{10}$, $\omega_{10} - \omega_{20} + \omega$
B6	$\omega = 7\omega_{10} = \frac{7}{3}\omega_{20}$	$\omega_{20} - 2\omega_{10}$, $\omega - 2\omega_{20}$	$3\omega_{10}$, $\omega - \omega_{10} - \omega_{20}$
B7	$\omega = 9\omega_{10} = 3\omega_{20}$	$\omega_{20} - 2\omega_{10}$	$3\omega_{10}$, $\omega - 2\omega_{20}$
B8	Non-Resonant Case	$\omega_{20} - 2\omega_{10}$	$3\omega_{10}$

Table 4.3 Classification of the external resonance
(internal resonance : $\omega_{20} = s\omega_{10}$ where $s \neq 1, 3$)

Type	Relations among ω_{10}, ω_{20}, and ω	$s = \frac{\omega_{20}}{\omega_{10}}$	Frequencies which are equal to ω_{10}	Frequencies which are equal to ω_{20}
C1	$\omega = \frac{\omega_{10}}{2} = \frac{\omega_{20}}{3}$	$\frac{3}{2}$	$\omega - \omega_{10} + \omega_{20}$	3ω, $2\omega_{10} - \omega$
C2	$\omega = \frac{\omega_{10}}{3} = \frac{\omega_{20}}{5}$	$\frac{5}{3}$	3ω, $\omega_{20} - 2\omega$, $\omega - \omega_{10} + \omega_{20}$	$\omega_{10} + 2\omega$, $2\omega_{10} - \omega$
C3	$\omega = 3\omega_{10} = \frac{3}{2}\omega_{20}$	2	$\omega - 2\omega_{10}$, $2\omega_{20} - \omega$	$\omega_{10} - \omega_{20} + \omega$
C4	$\omega = \frac{\omega_{10}}{3} = \frac{\omega_{20}}{7}$	$\frac{7}{3}$	3ω, $\omega_{20} - \omega_{10} - \omega$	$2\omega_{10} + \omega$
C5	$\omega = 3\omega_{10} = \frac{3}{5}\omega_{20}$	5	$\omega - 2\omega_{10}$, $2\omega - \omega_{20}$, $\omega_{20} - \omega_{10} - \omega$	$2\omega - \omega_{10}$, $2\omega_{10} + \omega$
C6	$\omega = 3\omega_{10} = \frac{3}{7}\omega_{20}$	7	$\omega - 2\omega_{10}$, $\omega_{20} - 2\omega$	$\omega_{10} + 2\omega$
C7	$\omega = 3\omega_{10} = \frac{\omega_{20}}{3}$	9	$\omega - 2\omega_{10}$	3ω

Table 4.4 Classification of the external resonance
(internal resonance : $\omega_{10} = \omega_{20}$)

Type	Relations among ω_{10}, ω_{20}, and ω	Frequencies which are equal to ω_{10} $(= \omega_{20})$
D1	$\omega = \omega_{10} = \omega_{20}$	ω_{10}, ω_{20}, ω; $2\omega_{10} - \omega_{20}$, $2\omega_{20} - \omega_{10}$
D2	$\omega = 3\omega_{10} = 3\omega_{20}$	ω_{10}, ω_{20}, $2\omega_{10} - \omega_{20}$, $2\omega_{20} - \omega_{10}$, $\omega - 2\omega_{10}$, $\omega - 2\omega_{20}$, $\omega - \omega_{10} - \omega_{20}$
D3	$\omega = \frac{\omega_{10}}{3} = \frac{\omega_{20}}{3}$	ω_{10}, ω_{20}, $2\omega_{10} - \omega_{20}$, $2\omega_{20} - \omega_{10}$, 3ω
D4	Non-Resonant Case	ω_{10}, ω_{20}, $2\omega_{10} - \omega_{20}$, $2\omega_{20} - \omega_{10}$

A3: When $\omega = \omega_{10}/3$, we obtain*

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - \frac{1}{3}A_1^3 \cos\theta_1]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2$$

$$r_1\dot{\theta}_1 = \mu(\omega_{11}r_1 + \frac{\omega_1^2}{24n_1}\frac{k_2}{k_2 - k_1}A_1^3 \sin\theta_1) \qquad (4.19)$$

$$r_2\dot{\theta}_2 = \mu\omega_{21}r_2$$

A4: When $\omega = \omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}A_1^3 \cos\theta_2]$$

$$r_1\dot{\theta}_1 = \mu\omega_{11}r_1 \qquad (4.20)$$

$$r_2\dot{\theta}_2 = \mu(\omega_{21}r_2 + \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2}A_1^3 \sin\theta_2)$$

A5: When $\omega = 3\omega_{10}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_1^2 \cos 3\theta_1]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2$$

$$r_1\dot{\theta}_1 = \mu(\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1 r_1^2 \sin 3\theta_1) \qquad (4.21)$$

$$r_2\dot{\theta}_2 = \mu\omega_{21}r_2$$

---

* We see from Eqs. (4.1) that $\omega_1 - 3\omega = \omega_1 - \omega_{10} = \mu\omega_{11} + O(\mu^2)$.

A6: When $\omega = 3\omega_{20}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - r_2^2 - 2r_2^2)r_2 - A_1 r_2^2 \cos 3\theta_2]$$

$$r_1\dot{\theta}_1 = \mu\omega_{11}r_1 \tag{4.22}$$

$$r_2\dot{\theta}_2 = \mu(\omega_{21}r_2 + \frac{\omega_2^2}{8n_1}\frac{k_2}{k_2 - k_2}A_1 r_2^2 \sin 3\theta_2)$$

A7: When $\omega = (\omega_{10} + \omega_{20})/2$, we obtain*

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1^2 r_2 \cos(\theta_1 + \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1^2 r_1 \cos(\theta_1 + \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1^2 r_2 \sin(\theta_1 + \theta_2)] \tag{4.23}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}A_1^2 r_1 \sin(\theta_1 + \theta_2)]$$

A8: When $\omega = (\omega_{20} - \omega_{10})/2$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1^2 r_2 \cos(\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1^2 r_1 \cos(\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1^2 r_2 \sin(\theta_1 - \theta_2)] \tag{4.24}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_1 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}A_1^2 r_1 \sin(\theta_1 - \theta_2)]$$

---

* We see from Eqs. (4.1) that $(\omega_1 + \omega_2)/2 - \omega = \mu(\omega_{11} + \omega_{21})/2 + O(\mu^2)$.

A9: When $\omega = \omega_{10} + 2\omega_{20}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_2^2 \cos(\theta_1 + 2\theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - 2A_1 r_1 r_2 \cos(\theta_1 + 2\theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1 r_2^2 \sin(\theta_1 + 2\theta_2)] \tag{4.25}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 + \frac{\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2}A_1 r_1 r_2 \sin(\theta_1 + 2\theta_2)]$$

A10: When $\omega = 2\omega_{20} - \omega_{10}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_2^2 \cos(\theta_1 - 2\theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - 2A_1 r_1 r_2 \cos(\theta_1 - 2\theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1 r_2^2 \sin(\theta_1 - 2\theta_2)] \tag{4.26}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2}A_1 r_1 r_2 \sin(\theta_1 - 2\theta_2)]$$

A11: When $\omega = 2\omega_{10} + \omega_{20}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - 2A_1 r_1 r_2 \cos(2\theta_1 + \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1 r_1^2 \cos(2\theta_1 + \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{4n_1}\frac{k_2}{k_2 - k_1}A_1 r_1 r_2 \sin(2\theta_1 + \theta_2)] \tag{4.27}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}A_1 r_1^2 \sin(2\theta_1 + \theta_2)]$$

A12: When $\omega = \pm (2\omega_{10} - \omega_{20})$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - 2A_1 r_1 r_2 \cos(2\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1 r_1^2 \cos(2\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{4n_1}\frac{k_2}{k_2 - k_1}A_1 r_1 r_2 \sin(2\theta_1 - \theta_2)] \tag{4.28}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}A_1 r_1^2 \sin(2\theta_1 - \theta_2)]$$

A13: When no external resonance occurs, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_1$$

$$\dot{\theta}_1 = \mu\omega_{11} \tag{4.29}$$

$$\dot{\theta}_2 = \mu\omega_{21}$$

It is easily seen that some of the averaged equations (4.12), (4.13), and (4.19) through (4.29), are derived from the rest of the equations by interchanging some parameters. For instance, Eqs. (4.13) are derived from Eqs. (4.12) by interchanging ω_{10} and ω_{20}, ω_{11} and ω_{21}, r_1 and r_2, θ_1 and θ_2, k_1 and k_2, ρ_1 and ρ_2. Equations (4.24) are also derived from Eqs. (4.23) by interchanging ω_{11} and $-\omega_{11}$, θ_1 and $-\theta_1$. Therefore, we treat only the autonomous systems (4.12), (4.19), (4.21), (4.23), (4.25), and (4.29) in the following analysis.

4.3 Entrained Oscillations in a System without Internal Resonance

As mentioned above, six types of autonomous equations are obtained. Equations (4.12) are derived when $\omega \cong \omega_1$. One may expect the occurrence of harmonic entrainment. Similarly, Eqs. (4.19) and (4.21) are derived when $\omega \cong \omega_1/3$ and $\omega \cong 3\omega_1$, respectively. In these cases one may expect the occurrence of higher-harmonic and subharmonic entrainments. Equations (4.23) and (4.25) are the autonomous systems when $\omega \cong (\omega_1 + \omega_2)/2$ and $\omega \cong \omega_1 + 2\omega_2$. Under these conditions a kind of resonance among ω_1, ω_2, and ω may occur. Equations (4. 29) are the system without external resonance. In this section, we investigate the harmonic, higher-harmonic, subharmonic, and combination oscillations as caused by frequency entrainment.

4.3.1 Harmonic Entrainment

We consider the entrainment of frequency, i.e., the external resonance which occurs when $\omega \cong \omega_1$.

(a) Steady-State Solutions

The steady-state solutions of Eqs. (4.12) are obtained by equating $\dot{r}_i = 0$ and $\dot{\theta}_i = 0$, i.e.,

$$
\begin{aligned}
&(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10} - \frac{4}{\omega}n_1 B_1 \sin\theta_{10} = 0 \\
&(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20} = 0 \\
&\omega_{11}r_{10} - \frac{\omega_1^2}{2}\,\frac{k_2}{k_2 - k_1}\,\frac{1}{\omega}B_1\cos\theta_{10} = 0 \\
&\omega_{21}r_{20} = 0
\end{aligned}
\tag{4.30}
$$

where r_{10}, r_{20}, and θ_{10} denote the steady-state values of r_1, r_2, and θ_1, respectively. We see, from Eqs. (4.30), that there are two different types of steady states, i.e.,

$$\begin{aligned}&(1) \quad r_{10} \neq 0, \quad r_{20} = 0\\&(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0\end{aligned} \tag{4.31}$$

In the steady state (1), eliminating θ_{10} leads to determination of the amplitude r_{10} as

$$[(\rho_1 - r_{10}^2)^2 + \sigma_1^2]r_{10}^2 = \left(\frac{4n_1B_1}{\omega}\right)^2 = \left(\frac{4n_1B}{\mu\omega}\right)^2 \tag{4.32}$$

where

$$\begin{aligned}\sigma_1 &= \frac{8n_1}{\omega_1^2}\,\frac{k_2 - k_1}{k_2}\,\omega_{11}\\&= \frac{8n_1}{\mu\omega_1^2}\,\frac{k_2 - k_1}{k_2}\,(\omega_1 - \omega) \quad : \quad \text{detuning}\end{aligned} \tag{4.33}$$

and the phase angle θ_{10} is given by

$$\begin{aligned}\sin\theta_{10} &= \frac{\omega}{4n_1B_1}(\rho_1 - r_{10}^2)r_{10}\\\cos\theta_{10} &= \frac{\omega}{4n_1B_1}\sigma_1 r_{10}\end{aligned} \tag{4.34}$$

In the steady state (2), eliminating θ_{10} from Eqs. (4.30) gives

$$\begin{aligned}&[(\rho_1 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_1^2]r_{10}^2 = \left(\frac{4n_1B_1}{\omega}\right)^2 = \left(\frac{4n_1B}{\mu\omega}\right)^2\\&\rho_2 - 2r_{10}^2 - r_{20}^2 = 0\end{aligned} \tag{4.35}$$

Eliminating r_{20}^2 from Eqs. (4.35) gives

$$[(\rho_1 - 2\rho_2 + 3r_{10}^2)^2 + \sigma_1^2]r_{10}^2 = \left(\frac{4n_1B_1}{\omega}\right)^2 \tag{4.36}$$

By using Eqs. (4.35) and (4.36), the amplitudes r_{10} and r_{20} are calculated. Regarding the frequencies, from Eqs. (4.30) we obtain $\omega_{21} = 0$. Therefore, from Eqs. (4.1) we obtain

$$\omega_{20} = \omega_2 \tag{4.37}$$

The phase angle θ_{10} may also be found to be

$$\begin{aligned} \sin\theta_{10} &= \frac{\omega}{4n_1B_1}(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10} \\ \cos\theta_{10} &= \frac{\omega}{4n_1B_1}\sigma_1 r_{10} \end{aligned} \tag{4.38}$$

From the solution (4.7), we see that the steady state (1) corresponds to the periodic oscillation of frequency ω, i.e., the harmonic entrainment. The steady state (2) corresponds to the combination oscillation having the components of frequencies ω and ω_2. This oscillation is generally almost periodic.

When the amplitude B_1 of the external force tends to zero, we obtain from Eqs. (4.32)

$$r_{10} = 0, \quad r_{20} = 0$$

or

$$r_{10}^2 = \rho_1, \; r_{20} = 0, \; \sigma_1 = 0 \tag{4.39}$$

and from Eqs. (4.35)

$$r_{10} = 0, \quad r_{20}^2 = \rho_2, \quad \omega_{21} = 0$$

or

$$r_{10}^2 = (2\rho_2 - \rho_1)/3, \quad r_{20}^2 = (2\rho_1 - \rho_2)/3, \quad \omega_{21} = 0, \quad \sigma_1 = 0 \tag{4.40}$$

Solutions (4.39) and (4.40) coincide with those of the self-excited oscillations given by Eqs. (2.11).

When $\omega \doteqdot \omega_2$, the solutions of Eqs. (4.13) are obtained by interchanging ω_{10} and ω_{20}, ω_{11} and ω_{21}, r_{10} and r_{20}, θ_{10} and θ_{20}, k_1 and k_2, ρ_1 and ρ_2 in the results of this section.

(b) Stability Investigation

The steady-state solutions given by Eqs. (4.32) through (4.38) are maintained only when they are stable. The stability of the solutions is tested by the behavior of small variations, ξ_1, ξ_2, η_1, and η_2, from the steady-state values, r_{10}, r_{20}, θ_{10}, and θ_{20}, respectively, i.e.,*

$$\begin{aligned} r_1 &= r_{10} + \xi_1, \qquad & r_2 &= r_{20} + \xi_2 \\ \theta_1 &= \theta_{10} + \eta_1, \qquad & \theta_2 &= \theta_{20} + \eta_2 \end{aligned} \tag{4.41}$$

If these variations tend to zero with increase of time t, the corresponding solution is stable. Substituting Eqs. (4.41) into Eqs. (4.12) gives the following variational equations.

$$\begin{aligned} \dot{\xi}_1 &= a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\eta_1 + a_{14}\eta_2 \\ \dot{\xi}_2 &= a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\eta_1 + a_{24}\eta_2 \\ \dot{\eta}_1 &= a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\eta_1 + a_{34}\eta_2 \\ \dot{\eta}_2 &= a_{41}\xi_1 + a_{42}\xi_2 + a_{43}\eta_1 + a_{44}\eta_2 \end{aligned} \tag{4.42}$$

where

$$a_{11} = \left(\frac{\partial h_1}{\partial r_1}\right)_0 = \frac{\mu\omega_1^2 k_2}{8n_1(k_2 - k_1)}\left(\rho_1 - 3r_{10}^2 - 2r_{20}^2\right)$$

$$a_{12} = \left(\frac{\partial h_1}{\partial r_2}\right)_0 = -\frac{\mu\omega_1^2 k_2}{2n_1(k_2 - k_1)} r_{10}r_{20}$$

$$a_{13} = \left(\frac{\partial h_1}{\partial \theta_1}\right)_0 = -\frac{\mu\omega_1^2 B_1}{2\,\omega\,(k_2 - k_1)} \cos\theta_{10}$$

$$a_{14} = \left(\frac{\partial h_1}{\partial \theta_2}\right)_0 = 0$$

---

* See Appendix II.

$$
\begin{aligned}
a_{21} &= \left(\frac{\partial h_2}{\partial r_1}\right)_0 = -\frac{\mu\omega_2^2 k_1}{2n_1(k_1 - k_2)}\, r_{10} r_{20} \\
a_{22} &= \left(\frac{\partial h_2}{\partial r_2}\right)_0 = \frac{\mu\omega_2^2 k_1}{8n_1(k_1 - k_2)}\, (\rho_2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= \left(\frac{\partial h_2}{\partial \theta_1}\right)_0 = 0 \\
a_{24} &= \left(\frac{\partial h_2}{\partial \theta_2}\right)_0 = 0 \\
a_{31} &= \left(\frac{\partial h_3}{\partial r_1}\right)_0 = \frac{\mu\omega_1^2 k_1}{2\,\omega\,(k_1 - k_2)}\, \frac{B_1}{r_{10}^2} \cos\theta_{10} \\
a_{32} &= \left(\frac{\partial h_3}{\partial r_2}\right)_0 = 0 \\
a_{33} &= \left(\frac{\partial h_3}{\partial \theta_1}\right)_0 = \frac{\mu\omega_1^2 k_1}{2\,\omega\,(k_1 - k_2)}\, \frac{B_1}{r_{10}} \sin\theta_{10} \\
a_{34} &= \left(\frac{\partial h_3}{\partial \theta_2}\right)_0 = 0 \\
a_{41} &= \left(\frac{\partial h_4}{\partial r_1}\right)_0 = 0 \\
a_{42} &= \left(\frac{\partial h_4}{\partial r_2}\right)_0 = 0 \\
a_{43} &= \left(\frac{\partial h_4}{\partial \theta_1}\right)_0 = 0 \\
a_{44} &= \left(\frac{\partial h_4}{\partial \theta_2}\right)_0 = 0
\end{aligned}
\tag{4.43}
$$

The symbol $(\;)_0$ denotes the insertion of the steady-state values, r_{10}, r_{20}, θ_{10}, and θ_{20} after differentiation. The characteristic equation of the system defined by Eqs. (4.42) is given by

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\lambda & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{vmatrix} = 0$$

or

$$\lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + s = 0 \tag{4.44}$$

where

$$p = -(a_{11} + a_{22} + a_{33} + a_{44})$$

$$q = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

$$r = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{4.45}$$

$$s = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The steady-state solution is stable, provided that the real parts of λ's are negative. The stability conditions are given by using the Routh-Hurwitz criterion (see Appendix II), i.e.,

$$\begin{gathered} p > 0, \quad r > 0, \quad s > 0, \\ pqr - r^2 - p^2 s > 0 \end{gathered} \tag{4.46}$$

As one sees from (4.43), $a_{4i} = 0$ $(i = 1,..,4)$, i.e., $s = 0$ in Eqs. (4.45). Therefore the characteristic equation (4.44) is reduced to

$$\lambda(\lambda^3 + p\lambda^2 + q\lambda + r) = 0 \tag{4.47}$$

Thus, one of the characteristic root is zero.* Hence we see that the solution is stable provided that †

$$p > 0, \quad r > 0, \quad pq - r > 0 \tag{4.48}$$

We consider the stability conditions for the two types of solutions in Eqs. (4.31).

(1) For $r_{10} \neq 0$, $r_{20} = 0$, we obtain the following characteristic equation by substituting Eqs. (4.43) into Eqs. (4.42) and using Eqs. (4.32) through (4.34), i.e.,

$$\lambda[\lambda - m_2(\rho_2 - 2r_{10}^2)]\left\{[\lambda - \mu m_1(\rho_1 - r_{10}^2)][\lambda - \mu m_1(\rho_1 - 3r_{10}^2)] + (\mu\omega_{11})^2\right\} = 0 \tag{4.49}$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

From the condition that the real parts of the roots λ's of Eqs. (4.49) are negative, we obtain

$$r_{10}^2 > \frac{1}{2}\rho_1, \qquad r_{10}^2 > \frac{1}{2}\rho_2$$

$$\frac{(r_{10}^2 - \frac{2}{3}\rho_1)^2}{(\frac{1}{3}\rho_1)^2} + \frac{\sigma_1^2}{(\frac{1}{\sqrt{3}}\rho_1)^2} > 1 \tag{4.50}$$

where σ_1 is given by Eq. (4.33).

(2) For $r_{10} \neq 0$, $r_{20} \neq 0$, the stability conditions are given by (4.48). By using Eqs. (4.43), (4.45), and (4.35) through (4.38), we obtain

---

* This result is due to the fact that the frequency ω_2 does not entrained by the driving frequency ω in the system (4.12).

† It is readily verified that the vertical tangency of the response curve occurs at the stability limit $r = 0$ (see Appendix II).

$$p = 2\mu\left\{-m_1\rho_1 + (2m_1 + m_2)\rho_2 - 2(m_1 + m_2)r_{10}^2\right\},$$

$$\begin{aligned} q &= \mu^2 m_1^2(\rho_1 - 2\rho_2 + r_{10}^2)(\rho_1 - 2\rho_2 + 3r_{10}^2) - 4\mu^2 m_1 m_2(\rho_2 - 2r_{10}^2) \\ &\quad \times(\rho_1 - 2\rho_2 + 6r_{10}^2) + (\mu\omega_{11})^2 \end{aligned} \tag{4.51}$$

$$r = 2\mu^3 m_2 r_{20}^2[m_1^2(\rho_1 - 2\rho_2 + 3r_{10}^2)(\rho_1 - 2\rho_2 + 6r_{10}^2) + \omega_{11}^2]$$

The first two conditions of (4.48) may be rewritten as

$$\begin{aligned} & r_{10}^2 < \frac{1}{2(m_1 + m_2)}[(2m_1 + m_2)\rho_2 - m_1\rho_1] \\ & \frac{[r_{10}^2 - \frac{2}{9}(2\rho_2 - \rho_1)]^2}{[\frac{1}{9}(2\rho_2 - \rho_1)]^2} + \frac{\sigma_1^2}{[\frac{1}{\sqrt{3}}(2\rho_2 - \rho_1)]^2} > 1 \end{aligned} \tag{4.52}$$

(c) Numerical Example

In this chapter a numerical analysis was carried out for the parameters as given by*

$$k = \sqrt{\chi_1\chi_2} = 0.5 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

The amplitudes r_{10}, r_{20}, and the phase angle θ_{10} are calculated by using Eqs. (4.32) through (4.38). Figure 4.2 shows the response curves for the harmonic and the combination oscillations. The harmonic oscillations corresponding to the steady state (1) of Eqs. (4.31) are represented by thick lines. On the other hand, the amplitudes r_{10}, r_{20}, and the phase angle θ_{10} of the combination oscillations are represented by fine lines. The stability of the oscillations is tested by using the conditions (4.50) and (4.48). The unstable portions of

---

* When no driving force is applied, this system has two kinds of self-excited oscillations, whose amplitudes are both $\sqrt{\rho_1} = \sqrt{\rho_2} = \sqrt{2}$ (see Sec. 2.3c).

the characteristic curves are shown by broken lines. Figure 4.3 shows the amplitude characteristics of the harmonic and the combination oscillations. The relationship between $(4n_1B/\mu\omega)^2$ and r_{10}^2, r_{20}^2 for several values of σ_1 is plotted.

In Fig. 4.4 are shown the regions of the harmonic entrainment and the combination oscillations on the $B\omega$ plane. When B is small and ω is in the neighborhood of ω_1, both the harmonic oscillation and the combination oscillation occur.*

4.3.2 Combination Oscillations without External Resonance

We see in Fig. 4.2a that there is no stable harmonic oscillation at all if the detuning σ_1 is sufficiently large compared with $(4n_1B/\mu\omega)^2$. In other words, the self-excited oscillations are not entrained by the driving frequency ω, if the driving frequency is far from both of the natural frequencies. In this case the solution is assumed as Eqs. (4.16). In this section we consider the case A13 where no external resonance occurs.†

The steady-state solutions of Eqs. (4.29) are obtained by equating $\dot{r}_1 = \dot{r}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$. Denoting the steady-state values of these variables by r_{10}, r_{20}, θ_{10}, and θ_{20}, respectively, we obtain

$$(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10} = 0$$

$$(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} = 0$$

---

* By considering the integral curves of the system (4.12), we can study the regions of initial conditions which lead to the harmonic oscillation and the combination oscillation. Such an analysis, however, is omitted in this study.

† If the frequencies ω_1, ω_2, and ω have a relationship shown in Table 4.1, the external resonance occurs.

$$\omega_{11}r_{10} = 0 \tag{4.53}$$

$$\omega_{21}r_{20} = 0$$

From the last two members of Eqs. (4.53) we obtain $\omega_{11} = 0$ and $\omega_{21} = 0$ when $r_{10} \neq 0$ and $r_{20} \neq 0$. This result shows that the components of frequencies ω_{10} and ω_{20} of the combination oscillations (4.16) coincide with the natural frequencies ω_1 and ω_2, respectively. We see, from Eqs. (4.53), that there are four different states of equilibrium, i.e.,

$$\begin{aligned}
&(1)\quad r_{10} = 0, \quad r_{20} = 0 \\
&(2)\quad r_{10}^2 = \rho_1 - 2A_1^2, \quad r_{20} = 0 \\
&(3)\quad r_{10} = 0, \quad r_{20}^2 = \rho_2 - 2A_1^2 \\
&(4)\quad r_{10}^2 = \frac{1}{3}(2\rho_2 - \rho_1 - 2A_1^2) \\
&\qquad\; r_{20}^2 = \frac{1}{3}(2\rho_1 - \rho_2 - 2A_1^2)
\end{aligned} \tag{4.54}$$

The entrained oscillation exists in (1); while, in (2), the solution has two frequency components ω_1 and ω. In (3), the solution has two frequency components ω_2 and ω. The solution in (4) has three frequency components ω_1, ω_2, and ω. Since ω_1, ω_2, and ω are generally imcommensurable, the solutions in (2), (3), and (4) are almost periodic.

Introducing small variations ξ_1 and ξ_2 defined by Eqs. (4.41), we obtain the variational equations of Eqs. (4.29), i.e.,

$$\begin{aligned}
\dot{\xi}_1 &= \mu m_1[(\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2)\xi_1 - 4r_{10}r_{20}\xi_2] \\
\dot{\xi}_2 &= \mu m_2[-4r_{10}r_{20}\xi_1 + (\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2)\xi_2]
\end{aligned} \tag{4.55}$$

The stability conditions are obtained from the condition that the real parts of the roots of the characteristic equation of the system (4.55) are negative.

(1) For $r_{10} = r_{20} = 0$, we obtain

$$A_1^2 > \frac{\rho_1}{2} \qquad \text{and} \qquad A_1^2 > \frac{\rho_2}{2} \tag{4.56}$$

(2) For $r_{10} \neq 0$, $r_{20} = 0$, we obtain

$$A_1^2 < \frac{\rho_1}{2} \qquad \text{and} \qquad A_1^2 < \rho_1 - \frac{\rho_2}{2} \tag{4.57}$$

(3) For $r_{10} = 0$, $r_{20} \neq 0$, we obtain

$$A_1^2 < \frac{\rho_2}{2} \qquad \text{and} \qquad A_1^2 < \rho_2 - \frac{\rho_1}{2} \tag{4.58}$$

(4) For $r_{10} \neq 0$, $r_{20} \neq 0$, we obtain

$$r_{10}^2 r_{20}^2 < 0 \tag{4.59}$$

Since both r_{10}^2 and r_{20}^2 are positive, condition (4.59) is not fulfilled.

From conditions (4.56), one sees that the harmonic entrainment occurs at any driving frequency ω provided the amplitude B of the external force is sufficiently large.

4.3.3 Higher-Harmonic Entrainment

We treat the case A3 in Table 4.1 which occurs when $\omega \cong \omega_1/3$. The steady-state solutions of Eqs. (4.19) are obtained by equating $\dot{r}_i = 0$ and $\dot{\theta}_i = 0$.

$$\begin{aligned}
&(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10} - \frac{1}{3} A_1^3 \cos \theta_{10} = 0 \\
&(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} = 0 \\
&\omega_{11} r_{10} + \frac{\omega_1^2}{24 n_1} \frac{k_2}{k_2 - k_1} A_1^3 \sin \theta_{10} = 0 \\
&\omega_{21} r_{20} = 0
\end{aligned} \tag{4.60}$$

We see, from (4.60), that there are two different states of equilibrium:

(1) $r_{10} = 0$, $r_{20} = 0$

$$(2) \quad r_{10} \neq 0,\ r_{20} \neq 0,\ \omega_{21} = 0 \tag{4.61}$$

In the steady state (1), eliminating θ_{10} gives

$$[(\rho_1 - 2A_1^2 - r_{10}^2)^2 + \sigma_3^2]r_{10}^2 = (\tfrac{1}{3} A_1^3)^2$$

where (4.62)

$$\sigma_3 = \frac{8n_1}{\omega_1^2}\frac{k_2 - k_1}{k_2}\omega_{11} = \frac{8n_1}{\mu\omega_1^2}\frac{k_2 - k_1}{k_2}(\omega_1 - 3\omega) \ : \text{detuning}$$

Then θ_{10} is given by

$$\sin\theta_{10} = -\frac{3\sigma_3 r_{10}}{A_1^3}$$

$$\cos\theta_{10} = \frac{3}{A_1^3}(\rho_1 - 2A_1^2 - r_{10}^2)r_{10} \tag{4.63}$$

In the steady state (2), eliminating θ_{10} in Eqs. (4.60) gives

$$[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_3^2]r_{10}^2 = (\tfrac{1}{3} A_1^3)^2$$

$$\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2 = 0 \tag{4.64}$$

Eliminating r_{10}^2 in Eqs. (4.64) gives

$$[(\rho_1 - 2\rho_2 + 2A_1^2 + 3r_{10})^2 + \sigma_3^2]r_{10}^2 = (\tfrac{1}{3} A_1^3)^2$$

then

$$r_{20}^2 = \rho_2 - 2A_1^2 - 2r_{10}^2$$

$$\sin\theta_{10} = -3\frac{r_{10}\sigma_3}{A_1^3} \tag{4.65}$$

$$\cos\theta_{10} = \frac{3}{A_1^3}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}$$

Since $\omega_{21} = 0$, the frequency ω_{20} is equal to ω_2. In the steady state (1), the oscillation consists of two frequency components ω and ω_{10} ($= 3\omega$). The natural frequency ω_1 is entrained by the third harmonic of driving frequency ω. The solution of the steady state (2) has the natural frequency ω_2 in addition to ω and 3ω. This oscillation is generally almost periodic.

The stability of the oscillations is tested in the same manner as we have done in Sec. 4.3.1b. The coefficients of the variational equations (4.42) are

$$
\begin{aligned}
a_{11} &= \mu m_1(\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2) \\
a_{12} &= - 4\mu m_1 r_{10} r_{20} \\
a_{13} &= \frac{1}{3}\mu m_1 A_1^3 \sin\theta_{10} \\
a_{14} &= 0 \\
a_{21} &= - 4\mu m_2 r_{10} r_{20} \\
a_{22} &= \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= a_{24} = 0 \\
a_{31} &= - \frac{1}{3}\mu m_1 \frac{A_1^3}{r_{10}^2} \sin\theta_{10} \\
a_{32} &= 0 \\
a_{33} &= \frac{1}{3}\mu m_1 \frac{A_1^3}{r_{10}} \cos\theta_{10} \\
a_{34} &= 0 \\
a_{4i} &= 0 \quad (i = 1, \ldots, 4)
\end{aligned}
\tag{4.66}
$$

where

$$
m_1 = \frac{\omega_1^2}{8n_1}\,\frac{k_2}{k_2 - k_1}, \qquad m_2 = \frac{\omega_2^2}{8n_1}\,\frac{k_1}{k_1 - k_2}
$$

The stability conditions are given by (4.48).* For the steady state (1), the

---

* It is seen that the vertical tangency of the response curve occurs at the stability limit $s = 0$ (see Appendix II).

characteristic equation of the variational equations of Eqs. (4.19) becomes

$$\lambda[\mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2) - \lambda]\left\{[\mu m_1(\rho_1 - 2A_1^2 - 3r_{10}^2) - \lambda]\right.$$
$$\left.\times[\mu m_1(\rho_1 - 2A_1^2 - r_{10}^2) - \lambda] + (\mu m_1\sigma_3)^2\right\} = 0 \tag{4.67}$$

Hence, the stability conditions are given by

$$\begin{aligned} r_{10}^2 &> (\rho_2 - 2A_1^2)/2 \\ r_{10}^2 &> (\rho_1 - 2A_1^2)/2 \\ \frac{\left\{r_{10}^2 - \frac{2}{3}(\rho_1 - 2A_1^2)\right\}^2}{\left(\frac{\rho_1 - 2A_1^2}{3}\right)^2} &+ \frac{\frac{2}{3}}{\left(\frac{\rho_1 - 2A_1^2}{\sqrt{3}}\right)^2} > 1 \end{aligned} \tag{4.68}$$

<u>Numerical Example</u>

A numerical analysis of the response characteristics for the higher-harmonic and the combination oscillations was carried out for the same values of the system parameters as those in Sec. 4.3.1c, i.e.,

$$k = 0.5 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

The response curves calculated by using Eqs. (4.62) through (4.65) are illustrated in Figs. 4.5 and 4.6. The thick lines in the figures represent the amplitude r_{10} and the phase angle θ_{10} of the higher-harmonic oscillation. The stability of the solutions is tested by using conditions (4.48) and Eqs. (4.66). The broken lines of the response curves show the unstable oscillations. The amplitudes r_{10}, r_{20}, and the phase angle θ_{10} of the combination oscillation are represented by fine lines in Figs. 4.5 and 4.6. It is seen that both the higher-harmonic oscillation and the combination oscillation occur for small values of σ_3.

4.3.4 Subharmonic Entrainment

We treat the case A5 in Table 4.1 which occurs when $\omega \cong 3\omega_1$. The steady-state solutions of Eqs. (4.21) are obtained by setting $\dot{r}_i = 0$ and $\dot{\theta}_i = 0$, i.e.,

$$\begin{aligned} &(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10} - A_1 r_{10}^2 \cos 3\theta_{10} = 0 \\ &(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} = 0 \\ &\omega_{11} r_{10} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} A_1 r_{10}^2 \sin 3\theta_{10} = 0 \\ &\omega_{21} r_{20} = 0 \end{aligned} \tag{4.69}$$

Equations (4.69) give four different states of equilibrium.

$$\begin{aligned} &(1)\quad r_{10} = 0, \quad r_{20} = 0 \\ &(2)\quad r_{10} \neq 0, \quad r_{20} = 0 \\ &(3)\quad r_{10} = 0, \quad r_{20} \neq 0, \quad \omega_{21} = 0 \\ &(4)\quad r_{10} \neq 0, \quad r_{20} \neq 0, \quad \omega_{21} = 0 \end{aligned} \tag{4.70}$$

In the steady state (2), eliminating θ_{10} gives

$$(\rho_1 - 2A_1^2 - r_{10}^2)^2 + \sigma_{1/3}^2 = A_1^2 r_{10}^2$$

where

$$\sigma_{1/3} = \frac{8n_1}{\omega_1^2}\frac{k_2 - k_1}{k_2}\omega_{11} = \frac{8n_1}{\mu\omega_1^2}\frac{k_2 - k_1}{k_2}(\omega_1 - \omega/3) \ : \ \text{detuning}$$

Then,

$$\begin{aligned} \sin 3\theta_{10} &= -\frac{\sigma_{1/3}}{A_1 r_{10}} \\ \cos 3\theta_{10} &= \frac{1}{A_1 r_{10}}(\rho_1 - 2A_1^2 - r_{10}^2) \end{aligned} \tag{4.71}$$

In the steady state (3), the amplitude r_{20} is given by

$$r_{20}^2 = \rho_2 - 2A_1^2 \tag{4.72}$$

From the relation $\omega_{21} = 0$, the frequency ω_{20} is found to be equal to ω_2. In the steady state (4), eliminating θ_{10} gives

$$\left.\begin{aligned} &(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_{1/3}^2 = A_1^2 r_{10}^2 \\ &\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2 = 0 \end{aligned}\right\} \tag{4.73}$$

Eliminating r_{20}^2 in Eqs. (4.73) gives

$$(\rho_1 - 2\rho_2 + 2A_1^2 + 3r_{10}^2)^2 + \sigma_{1/3}^2 = A_1^2 r_{10}^2$$

Then

$$\left.\begin{aligned} &r_{20}^2 = \rho_2 - 2A_1^2 - 2r_{10}^2 \\ &\sin 3\theta_{10} = -\frac{1}{A_1 r_{10}} \sigma_{1/3} \\ &\cos 3\theta_{10} = \frac{1}{A_1 r_{10}} (\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) \end{aligned}\right\} \tag{4.74}$$

The stability conditions are derived from the characteristic equation (4.47). The results are as follows.

(1) For $r_{10} = r_{20} = 0$,

$$A_1^2 > \frac{\rho_1}{2} \quad \text{and} \quad A_1^2 > \frac{\rho_2}{2} \tag{4.75}$$

(2) For $r_{10} \neq 0$, $r_{20} = 0$,

$$r_{10}^2 > \frac{\rho_1}{2} - A_1^2, \quad r_{10}^2 > \frac{\rho_2}{2} - A_1^2, \quad \text{and} \quad r_{10}^2 > \rho_1 - \frac{3}{2} A_1^2 \tag{4.76}$$

(3) For $r_{10} = 0$, $r_{20} \neq 0$,

$$A_1^2 < \frac{\rho_2}{2} \quad \text{and} \quad A_1^2 < \rho_2 - \frac{\rho_1}{2} \tag{4.77}$$

(4) For $r_{10} \neq 0$, $r_{20} \neq 0$, the stability conditions are given by (4.48). By using Eqs. (4.74) it is shown that if r in Eqs. (4.47) is positive, q becomes

negative. The stability conditions (4.48) are never fulfilled. Therefore, this solution is unstable.

In the steady state (1) the harmonic entrainment occurs. The steady state (2) corresponds to the subharmonic entrainment. The self-excited oscillation having the frequency ω_1 is entrained by a frequency which is one-third of the driving frequency ω. Therefore, the resulting oscillation consists of two frequency components ω and ω_{10} ($= \omega/3$). The steady state (3) corresponds to the combination oscillation having the frequency components ω and ω_2. In the steady state (4) the combination oscillation having the frequency components ω, $\omega/3$, and ω_2 occurs. The first and third cases are identical with those already discussed in (1) and (3) of Eqs. (4.54), respectively. Hence, the stability conditions given by (4.75) and (4.77) are also identical with (4.56) and (4.58).

Numerical Example

A numerical analysis of the response characteristics for the subharmonic and the combination oscillations corresponding respectively to (2) and (4) of Eqs. (4.70) was carried out by using the same values of parameters as in the preceding sections, i.e.,

$$k = 0.5 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

The response characteristics calculated by using Eqs. (4.71) and (4.74) are illustrated in Figs. 4.7 and 4.8. The regions of the harmonic and 1/3 harmonic entrainments given by the conditions (4.75) and (4.76) have an overlapping area on the $B\omega$ plane (see Sec. 4.4).* Therefore, if the driving frequency ω is kept constant and the amplitude B is varied, the response curves are subject to hysteresis as shown in Fig. 4.8.

4.3.5 Frequency Entrainment Which Occurs When $\omega \cong (\omega_1 + \omega_2)/2$

The frequency entrainment discussed in the preceding sections occurs between the driving frequency and one of the natural frequencies. The occurrence of these phenomena may be expected from the study of a system with one degree of freedom. In a system with two degrees of freedom a kind of resonance occurs when the driving frequency ω is in the neighborhood of a linear combination of the two natural frequencies ω_1 and ω_2. In this section we treat the case A7 where $\omega \cong (\omega_1 + \omega_2)/2$.

The steady-state oscillations of Eqs. (4.23) are obtained by equating $\dot{r}_1 = \dot{r}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$.

$$
\begin{aligned}
&(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10} - A_1^2 r_{20} \cos(\theta_{10} + \theta_{20}) = 0 \\
&(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} - A_1^2 r_{10} \cos(\theta_{10} + \theta_{20}) = 0 \\
&\omega_{11} r_{10} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} A_1^2 r_{20} \sin(\theta_{10} + \theta_{20}) = 0 \\
&\omega_{21} r_{20} + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2} A_1^2 r_{10} \sin(\theta_{10} + \theta_{20}) = 0
\end{aligned}
\tag{4.78}
$$

We see, from Eqs. (4.78), that there are two types of steady states.

$$
\begin{aligned}
&(1) \quad r_{10} = 0, \quad r_{20} = 0 \\
&(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0
\end{aligned}
\tag{4.79}
$$

In the steady state (1), the harmonic entrainment occurs. In the steady state (2), the combination oscillation occurs. This oscillation has three frequency

---

* It depends on the initial condition as regards which kind of oscillations occurs in this area. By considering the integral curves of Eqs. (4.21), we can study the regions of initial conditions. Such an analysis, however, is omitted in this study.

components ω, ω_{10}, and ω_{20}, related by $\omega = (\omega_{10} + \omega_{20})/2$ and is generally almost periodic. Since both ω_{10} and ω_{20} are unknown, ω_{11} and ω_{21} in Eqs. (4.78) are also unknown; while the sum $\omega_{11} + \omega_{21}$ is calculated from Eqs. (4.1), i.e.,

$$\mu(\omega_{11} + \omega_{21}) = \omega_1 - \omega_{10} + \omega_2 - \omega_{20} + O(\mu^2) = \omega_1 + \omega_2 - 2\omega + O(\mu^2) \qquad (4.80)$$

Eliminating cos $(\theta_{10} + \theta_{20})$ from the first two members of Eqs. (4.78) gives

$$(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2 = 0 \qquad (4.81)$$

By using Eqs. (4.80) and eliminating $\theta_{10} + \theta_{20}$ from Eqs. (4.78), we obtain

$$\begin{aligned}&[m_1(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) + m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)]^2 \\ &+ (\omega_{11} + \omega_{21})^2 = A_1^4\left(m_1 \frac{r_{20}}{r_{10}} + m_2 \frac{r_{10}}{r_{20}}\right)^2\end{aligned} \qquad (4.82)$$

where

$$m_1 = \frac{\omega_1^2}{8n_1} \frac{k_2}{k_2 - k_1}, \qquad m_2 = \frac{\omega_2^2}{8n_1} \frac{k_1}{k_1 - k_2}$$

Solving Eqs. (4.81) and Eqs. (4.82) simultaneously gives the amplitudes r_{10} and r_{20}. The phase angle $\theta_{10} + \theta_{20}$ is given by

$$\begin{aligned}\sin(\theta_{10} + \theta_{20}) &= -\frac{(\omega_{11} + \omega_{21})r_{10}r_{20}}{A_1^2(m_1 r_{20}^2 + m_2 r_{10}^2)} \\ \cos(\theta_{10} + \theta_{20}) &= \frac{r_{10}}{A_1^2 r_{20}}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) \\ &= \frac{r_{20}}{A_1^2 r_{10}}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)\end{aligned} \qquad (4.83)$$

The frequencies ω_{11} and ω_{21} in Eqs. (4.1) are obtained by substituting Eqs. (4.83) into the last two members of Eqs. (4.78).

$$\omega_{11} = \frac{m_1(\omega_{11} + \omega_{21})r_{20}^2}{m_1 r_{20}^2 + m_2 r_{10}^2}, \qquad \omega_{21} = \frac{m_2(\omega_{11} + \omega_{21})r_{10}^2}{m_1 r_{20}^2 + m_2 r_{10}^2} \qquad (4.84)$$

The frequencies ω_{10} and ω_{20} are determined from Eqs. (4.1), i.e.,

$$\begin{aligned} \omega_{10} &= \omega_1 - \mu\omega_{11} \\ \omega_{20} &= \omega_2 - \mu\omega_{21} \end{aligned} \tag{4.85}$$

The variational equations of Eqs. (4.23) are given by Eqs. (4.42). For the steady state (2) of Eqs. (4.79) the coefficients of Eqs. (4.42) are as follows:

$$\begin{aligned}
a_{11} &= \mu m_1(\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2) \\
a_{12} &= -\mu m_1[4r_{10}r_{20} + A_1^2 \cos(\theta_{10} + \theta_{20})] \\
a_{13} &= a_{14} = \mu m_1 A_1^2 r_{20} \sin(\theta_{10} + \theta_{20}) \\
a_{21} &= -\mu m_2[4r_{10}r_{20} + A_1^2 \cos(\theta_{10} + \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= a_{24} = \mu m_2 A_1^2 r_{10} \sin(\theta_{10} + \theta_{20}) \\
a_{31} &= -\mu m_1 A_1^2 \frac{r_{20}}{r_{10}^2} \sin(\theta_{10} + \theta_{20}) \\
a_{32} &= \mu m_1 \frac{A_1^2}{r_{10}} \sin(\theta_{10} + \theta_{20}) \\
a_{33} &= a_{34} = \mu m_1 A_1^2 \frac{r_{20}}{r_{10}} \cos(\theta_{10} + \theta_{20}) \\
a_{41} &= \mu m_2 \frac{A_1^2}{r_{20}} \sin(\theta_{10} + \theta_{20}) \\
a_{42} &= -\mu m_2 A_1^2 \frac{r_{10}}{r_{20}^2} \sin(\theta_{10} + \theta_{20}) \\
a_{43} &= a_{44} = \mu m_2 A_1 \frac{r_{10}}{r_{20}} \cos(\theta_{10} + \theta_{20})
\end{aligned} \tag{4.86}$$

Since $a_{i3} = a_{i4}$ $(i = 1,.., 4)$, one of the characteristic roots is zero. The stability conditions are also given by (4.48).

To investigate the stability of the steady state, $r_{10} = 0$, $r_{20} = 0$, we consider the small variations*:

$$x = \xi_1 \cos (\omega_{10}t + \theta_1), \quad y = \xi_2 \cos (\omega_{20}t + \theta_2) \tag{4.87}$$

If ξ_1 and ξ_2 approach zero with the lapse of time t, the solution $r_{10} = 0$, $r_{20} = 0$ is stable. The amplitudes ξ_i and the phase angles θ_i of Eqs. (4.87) satisfy the averaged equations (4.23). Replacing r_1 and r_2 in Eqs. (4.23) by ξ_1 and ξ_2, we obtain

$$\begin{aligned}
\dot{\xi}_1 &= \mu m_1[(\rho_1 - 2A_1^2)\xi_1 - A_1^2\xi_2 \cos (\theta_1 + \theta_2)] \\
\dot{\xi}_2 &= \mu m_2[(\rho_2 - 2A_1^2)\xi_2 - A_1^2\xi_1 \cos (\theta_1 + \theta_2)] \\
\xi_1\dot{\theta}_1 &= \mu[\omega_{11}\xi_1 + m_1A_1^2\xi_2 \sin (\theta_1 + \theta_2)] \\
\xi_2\dot{\theta}_2 &= \mu[\omega_{21}\xi_2 + m_2A_1^2\xi_1 \sin (\theta_1 + \theta_2)]
\end{aligned} \tag{4.88}$$

where

$$m_1 = \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}, \qquad m_2 = \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}$$

We introduce new variables a_1, b_1, a_2, and b_2 defined by

$$\begin{aligned}
a_1 &= \xi_1 \cos \theta_1, \qquad b_1 = \xi_1 \sin \theta_1 \\
a_2 &= \xi_2 \cos \theta_2, \qquad b_2 = \xi_2 \sin \theta_2
\end{aligned} \tag{4.89}$$

Substituting Eqs. (4.89) into Eqs. (4.88) yields the variational equations with

---

* The variational equations (4.42) of Eqs. (4.23) have the coefficients containing $1/r_{10}$, $1/r_{20}$, and $\theta_{10} + \theta_{20}$ as shown in Eqs. (4.86). In the steady state (1), however, the amplitudes r_{10} and r_{20} are zero and $\theta_{10} + \theta_{20}$ is unfixed. Therefore, the variational equations (4.42) are not applicable to the investigation of stability.

respect to a_i and b_i $(i = 1, 2)$. Then, the characteristic equation becomes

$$\begin{vmatrix} \mu m_1(\rho_1 - 2A_1^2) - \lambda & -\mu\omega_{11} & -\mu m_1 A_1^2 & 0 \\ \mu\omega_{11} & \mu m_1(\rho_1 - 2A_1^2) - \lambda & 0 & \mu m_1 A_1^2 \\ -\mu m_2 A_1^2 & 0 & \mu m_2(\rho_2 - 2A_1^2) - \lambda & -\mu\omega_{21} \\ 0 & \mu m_2 A_1^2 & \mu\omega_{21} & \mu m_2(\rho_2 - 2A_1^2) - \lambda \end{vmatrix}$$

$$\begin{aligned} = &\Big\{\lambda^2 - \mu[m_1(\rho_1 - 2A_1^2) + m_2(\rho_2 - 2A_1^2) + (\omega_{11} - \omega_{21})i]\lambda \\ &+ \mu^2[m_1 m_2(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) + \omega_{11}\omega_{21} - m_1 m_2 A_1^4] \\ &+ \mu^2 i[\omega_{11} m_2(\rho_2 - 2A_1^2) - \omega_{21} m_1(\rho_1 - 2A_1^2)]\Big\} \\ &\times\Big\{\lambda^2 - \mu[m_1(\rho_1 - 2A_1^2) + m_2(\rho_2 - 2A_1^2) - (\omega_{11} - \omega_{21})i]\lambda \\ &+ \mu^2[m_1 m_2(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) + \omega_{11}\omega_{21} - m_1 m_2 A_1^4] \\ &- \mu^2 i[\omega_{11} m_2(\rho_2 - 2A_1^2) - \omega_{21} m_1(\rho_1 - 2A_1^2)]\Big\} \\ = &\ 0 \end{aligned} \tag{4.90}$$

where $i = \sqrt{-1}$.

The conditions for stability are that the real parts of the characteristic roots λ's of Eq. (4.90) are negative. From Eq. (4.90) we obtain the stability conditions for the steady state (1) of Eqs. (4.79)*, i.e.,

---

* Let us consider a quadratic equation

$$\lambda^2 + (\ell + i\ell')\lambda + m + im' = 0$$

where ℓ, ℓ', m, and m' are real and $i = \sqrt{-1}$. After some algebraic manipulation, it is shown that the real parts of the roots λ's are negative provided that

$$\ell > 0 \qquad \text{and} \qquad \ell^2 m + \ell\ell' m' - m'^2 > 0$$

$$m_1(\rho_1 - 2A_1^2) + m_2(\rho_2 - 2A_1^2) < 0$$

and (4.91)

$$[m_1(\rho_1 - 2A_1^2) + m_2(\rho_2 - 2A_1^2)]^2[(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) - A_1^4]$$
$$+ (\omega_{11} + \omega_{21})^2(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) > 0$$

Especially, when $\rho_1 = \rho_2$ (i.e., $n_1 = n_2$), the conditions (4.91) are written as

$$A_1^2 > \frac{\rho_1}{2}$$

and (4.92)

$$\frac{(A_1^2 - \frac{2}{3}\rho_1)^2}{(\frac{\rho_1}{3})^2} + \frac{\left(\frac{\omega_{11} + \omega_{21}}{m_1 + m_2}\right)^2}{(\frac{\rho_1}{\sqrt{3}})^2} > 1$$

As mentioned in Sec. 4.3.2, the stability conditions for the harmonic oscillation are given by (4.56), provided that ω is not in the neighborhood of ω_1 and ω_2. The conditions (4.91) are, on the other hand, obtained when $\omega \cong (\omega_1 + \omega_2)/2$. Therefore, one may expect that the region of harmonic entrainment on the $B\omega$ plane contracts by the second condition of (4.91) when $\omega \cong (\omega_1 + \omega_2)/2$ (Sec. 4.4).

We consider the relationship among the system parameters when the combination oscillation disappears. The amplitudes r_{10} and r_{20} of the combination oscillation are given by Eqs. (4.81) and (4.82). When r_{10} and r_{20} tend to zero in Eq. (4.81), we obtain

$$\lim_{r_{10}, r_{20} \to 0} \left(\frac{r_{20}}{r_{10}}\right)^2 = \frac{\rho_1 - 2A_1^2}{\rho_2 - 2A_1^2} \tag{4.93}$$

When both r_{10} and r_{20} tend to zero, we obtain from Eqs. (4.82) and (4.93)

$$[m_1(\rho_1 - 2A_1^2) + m_2(\rho_2 - 2A_1^2)]^2[(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) - A_1^4]$$
$$+ (\omega_{11} + \omega_{21})^2(\rho_1 - 2A_1^2)(\rho_2 - 2A_1^2) = 0 \tag{4.94}$$

It is to be noted that Eq. (4.94) is identical with the stability limit which is obtained from the second condition of (4.91).

Numerical Example

Let us consider the same system parameters as before, i.e.,

$$\mu = 0.1 \qquad k = 0.5 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

We introduce the detuning σ defined by

$$\sigma = \frac{\omega_{11} + \omega_{21}}{m_1 + m_2} = \frac{8n_1(k_2 - k_1)}{\mu(\omega_1^2 k_2 - \omega_2^2 k_1)}(\omega_1 + \omega_2 - 2\omega) \tag{4.95}$$

By using Eqs. (4.81) to (4.83), the relations between σ and r_{10}^2, r_{20}^2, $\theta_{10} + \theta_{20}$ are illustrated for several values of A_1^2 in Fig. 4.9a and b. The frequencies ω_{10} and ω_{20} are calculated by using Eqs. (4.84) and (4.85) and plotted in Fig. 4.9c for $A_1^2 = 0.5$ (thick line) and 0.7 (fine line). Response curves for varying A_1 (i.e., B) are shown in Fig. 4.10.

The four points A to D in Figs. 4.9 and 4.10 show the amplitudes, phase angles, and frequencies of the four steady-state solutions for $A_1^2 = 0.7$ and $\sigma = 0.6$. It is seen from Fig. 4.9 that the values of r_{10}^2, r_{20}^2, and ω_{10} tend to $2 - 2A_1^2$, 0, and ω_1, respectively, in certain branches of the characteristic curves as the detuning $|\sigma|$ increases; while r_{10}^2, r_{20}^2, and ω_{20} $(= 2\omega - \omega_{10})$ tend to 0, $2 - 2A_1^2$, and ω_2, respectively, in the other branches. Therefore, the combination oscillations discussed in this section tend to those of non-resonant case [(2) and (3) of Eqs. (4.54)] as $|\sigma|$ increases. These results mean that there are no abrupt changes in the amplitudes and frequencies of combination oscillations between those having three frequency components and those having two frequency components.

4.3.6 Frequency Entrainment Which Occurs When $\omega \cong \omega_1 + 2\omega_2$

We discuss the case A9 where $\omega = \omega_{10} + 2\omega_{20}$. The steady-state solutions of Eqs. (4.25) are obtained by equating $\dot{r}_1 = \dot{r}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$, i.e.,

$$
\begin{aligned}
&(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10} - A_1 r_{20}^2 \cos(\theta_{10} + 2\theta_{20}) = 0 \\
&(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} - 2A_1 r_{10} r_{20} \cos(\theta_{10} + 2\theta_{20}) = 0 \\
&\omega_{11} r_{10} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} A_1 r_{20}^2 \sin(\theta_{10} + 2\theta_{20}) = 0 \\
&\omega_{21} r_{20} + \frac{\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2} A_1 r_{10} r_{20} \sin(\theta_{10} + 2\theta_{20}) = 0
\end{aligned}
\tag{4.96}
$$

From Eqs. (4.96), we consider the following three cases:

$$
\begin{aligned}
&(1)\quad r_{10} = 0, \quad r_{20} = 0 \\
&(2)\quad r_{10}^2 = \rho_1 - 2A_1^2, \quad r_{20} = 0, \quad \omega_{11} = 0 \\
&(3)\quad r_{10} \neq 0, \quad r_{20} \neq 0
\end{aligned}
\tag{4.97}
$$

The steady state (1) corresponds to the harmonic entrainment. The steady state (2) corresponds to the combination oscillation having the frequencies ω and ω_1. These cases have already been discussed in Sec. 4.3.2. The steady state (3) is also the combination oscillation, which has three frequency components ω, ω_{10}, and ω_{20}, related by $\omega = \omega_{10} + 2\omega_{20}$. By using Eqs. (4.1), we have

$$
\begin{aligned}
\mu(\omega_{11} + 2\omega_{21}) &= \omega_1 - \omega_{10} + 2\omega_2 - 2\omega_{20} + O(\mu^2) \\
&= \omega_1 + 2\omega_2 - \omega + O(\mu^2)
\end{aligned}
\tag{4.98}
$$

Eliminating $\cos(\theta_{10} + 2\theta_{20})$ from the first two members of Eqs. (4.96) gives

$$
2(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2 = 0 \tag{4.99}
$$

Eliminating $\theta_{10} + 2\theta_{20}$ from Eqs. (4.96) yields

$$\left\{[m_1(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) + 2m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)]^2 + (\omega_{11} + 2\omega_{21})^2\right\} r_{10}^2 = A_1^2(4m_2 r_{10}^2 + m_1 r_{20}^2)^2 \tag{4.100}$$

where

$$m_1 = \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}, \qquad m_2 = \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}$$

Solving Eqs. (4.98), (4.99), and (4.100) simultaneously gives the amplitudes r_{10} and r_{20}. The phase angle $\theta_{10} + 2\theta_{20}$ is given by

$$\begin{aligned} \sin(\theta_{10} + 2\theta_{20}) &= \frac{-(\omega_{11} + 2\omega_{21})r_{10}}{A_1(4m_2 r_{10}^2 + m_1 r_{20}^2)} \\ \cos(\theta_{10} + 2\theta_{20}) &= \frac{r_{10}}{A_1 r_{20}^2}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) \\ &= \frac{1}{2A_1 r_{10}}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2) \end{aligned} \tag{4.101}$$

The frequencies ω_{10} and ω_{20} are determined from Eqs. (4.1), (4.96), and (4.101), i.e.,

$$\begin{aligned} \omega_{10} &= \omega_1 - \mu\omega_{11} = \omega_1 + \mu m_1 A_1 \frac{r_{20}^2}{r_{10}} \sin(\theta_{10} + 2\theta_{20}) \\ \omega_{20} &= \omega_2 - \mu\omega_{21} = \omega_2 + 2\mu m_2 A_1 r_{10} \sin(\theta_{10} + 2\theta_{20}) \end{aligned} \tag{4.102}$$

The stability conditions are given by (4.48), where the coefficients a_{ij} ($i, j = 1,.., 4$) of the variational equations (4.42) are as follows:

$$\begin{aligned} a_{11} &= \mu m_1(\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2) \\ a_{12} &= -2\mu m_1[2r_{10}r_{20} + A_1 r_{20}\cos(\theta_{10}+2\theta_{20})] \\ a_{13} &= \frac{1}{2}a_{14} = \mu m_1 A_1 r_{20}^2 \sin(\theta_{10} + 2\theta_{20}) \\ a_{21} &= -2\mu m_2[2r_{10}r_{20} + A_1 r_{20}\cos(\theta_{10}+2\theta_{20})] \\ a_{22} &= \mu m_2[\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2 - 2A_1 r_{10}\cos(\theta_{10}+2\theta_{20})] \end{aligned}$$

$$
\begin{aligned}
a_{23} &= \frac{1}{2}\, a_{24} = 2\mu m_2 A_1 r_{10} r_{20} \sin(\theta_{10} + 2\theta_{20}) \\
a_{31} &= -\mu m_1 A_1 \frac{r_{20}^2}{r_{10}^2} \sin(\theta_{10} + 2\theta_{20}) \\
a_{32} &= 2\mu m_1 A_1 \frac{r_{20}}{r_{10}} \sin(\theta_{10} + 2\theta_{20}) \\
a_{33} &= \frac{1}{2}\, a_{34} = \mu m_1 A_1 \frac{r_{20}^2}{r_{10}} \cos(\theta_{10} + 2\theta_{20}) \\
a_{41} &= 2\mu m_2 A_1 \sin(\theta_{10} + 2\theta_{20}) \\
a_{42} &= 0 \\
a_{43} &= \frac{1}{2}\, a_{44} = 2\mu m_2 A_1 r_{10} \cos(\theta_{10} + 2\theta_{20})
\end{aligned}
\tag{4.103}
$$

Numerical Example

The numerical analysis was carried out for the same values of the system parameters as in the preceding sections, i.e.,

$$\mu = 0.1 \qquad k = 0.5 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

The response characteristics of the combination oscillations which correspond to the steady state (3) of Eqs. (4.97) are calculated by using Eqs. (4.98) to (4.102) and illustrated in Figs. 4.11 and 4.12. The detuning σ is defined by

$$\sigma = \frac{\omega_{11} + 2\omega_{21}}{m_1 + 2m_2} = \frac{\omega_1 + 2\omega_2 - \omega}{\mu(m_1 + 2m_2)} \tag{4.104}$$

The hysteresis of the response curves is observed in Fig. 4.12. As the detuning $|\sigma|$ increases, the amplitudes r_{10}^2, r_{20}^2, and the frequency ω_{20} tend to zero, $2 - 2A_1^2$, and ω_2, respectively. Therefore the combination oscillation having three frequency components tends to that having two frequency components ω and ω_2 [see Eqs. (4.54)].

4.4 Regions of Frequency Entrainment

In the preceding sections, we have investigated the entrained oscillations, i.e., the harmonic, higher-harmonic, subharmonic, and combination oscillations. The response characteristics of the entrained oscillations are obtained. The stability of these oscillations has been investigated by making use of the Routh-Hurwitz criterion. From the results obtained in the preceding sections, the regions of the frequency entrainment are produced on the $B\omega$ plane for

$$\mu = 0.1 \qquad k = 0.5 \qquad (n_2/n_1)^2\hat{\delta} = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

as illustrated in Fig. 4.13.* If the amplitude and the frequency of the external force are given inside these regions, the harmonic, higher-harmonic, subharmonic, or combination oscillation occurs. The higher-harmonic, subharmonic, or combination oscillation occurs within a narrow range of the driving frequency ω. On the other hand, the harmonic entrainment occurs at any driving frequency ω provided the amplitude B of the external force is sufficiently large.

From the results in the preceding sections, the stability conditions for the harmonic oscillation are given by [condition (4.56)] †

$$\begin{aligned} D_1 &\equiv \rho_1/2 - A_1^2 < 0 \\ D_2 &\equiv \rho_2/2 - A_1^2 < 0 \end{aligned} \tag{4.105}$$

---

* As mentioned before, the assumption of the solution (4.16) is not sufficient when $\omega \cong n_2$ (see the footnote of p. 43). The region of the harmonic entrainment for $\omega \cong n_2$ are calculated from the results in Sec. 6.3 which are obtained by assuming the solution as Eqs. (6.2).

† In this chapter, we consider the case where $n_1 = n_2$. Therefore, $\rho_1 = \rho_2$ [see Eqs. (2.8)] and $D_1 = D_2$.

provided that the driving frequency ω is not in the neighborhood of ω_1, ω_2, and $(\omega_2 \pm \omega_1)/2$. In Fig. 4.13, one sees that the continuity of the boundary curve $D_1 \equiv D_2 = 0$ is disturbed by the intrusion of the region of the combination oscillation at $\omega \cong (\omega_2 \pm \omega_1)/2$. On the other hand, the regions of the harmonic oscillation and 1/3-harmonic oscillation have an overlapping area. In this area common to the two regions, both the harmonic and 1/3-harmonic oscillations are stably sustained. The regions of the harmonic oscillation and the combination oscillation which occurs when $\omega \cong 2\omega_2 \pm \omega_1$ or $\omega \cong 2\omega_1 \pm \omega_2$ have also an overlapping area of this type. The boundary curve for the higher-harmonic entrainment is obtained from the stability conditions (4.68). This boundary curve tends to that given by $D_1 = 0$, as $|\sigma_3|$ becomes larger. It should be noted that there are no abrupt changes in the amplitudes of the harmonic and higher-harmonic components of an oscillation when B is varied across the regions of the harmonic and higher-harmonic entrainments [10].

As mentioned in Secs. 4.3.5 and 4.3.6, the combination oscillations having three frequency components occur when ω is in the neighborhood of $(\omega_2 \pm \omega_1)/2$, $2\omega_2 \pm \omega_1$, and $2\omega_1 \pm \omega_2$. There are no abrupt changes in the amplitudes of the frequency components of the oscillation when ω is varied across the region indicated by broken lines (cf. Figs. 4.9 and 4.11).

As mentioned in Secs. 4.3.1 and 4.3.3 in the regions of the harmonic and higher-harmonic entrainments, the combination oscillations may also occur for small values of B (see Fig. 4.4). The regions in which such combination oscillations occur are not illustrated in Fig. 4.13.

4.5 Concluding Remarks

The steady-state oscillations in a self-oscillatory system under a periodic force have been discussed. The autonomous systems which are derived by making use of the averaging method are classified according to the relationship among the driving frequency and the two natural frequencies. Then, we have investigated the phenomenon of frequency entrainment in the system without internal resonance. The response characteristics of the entrained oscillations are obtained.

From these results the regions of the frequency entrainment are reproduced on the $B\omega$ plane. If the amplitude and the frequency of the external force are properly chosen in these regions, the entrainment occurs at the harmonic, higher-harmonic, or subharmonic frequency of the external force. Moreover, if the driving frequency is in the neighborhood of a frequency which is a linear combination of two natural frequencies, this linear combination of the natural frequencies are entrained by the driving frequency. In this case the resulting oscillation is an almost periodic and consists of three harmonic components. In the region other than these regions of frequency entrainment, there are two kinds of combination oscillations which consist of two simple harmonic components, one with the driving frequency and the other with one of the two natural frequencies (Sec. 4.4a).

If the external force is prescribed close to the regions of entrainment, the analysis in this chapter does not account for the almost periodic oscillation very well. In this case, the waveform of the almost periodic oscillation differs considerably from that obtained as a sum of two or three simple harmonic oscillations. The almost periodic oscillations of this type are omitted in this study.*

By making use of the results in the preceding sections, the types of the external resonances and the frequency components contained in the steady-state oscillations are summarized in Table 4.5. The symbols in Table 4.5 correspond to those in Table 4.1.

* This almost periodic oscillation is considered to develop from the entrained oscillation and may be expressed by a sum of the forced and free oscillations, but the amplitude and phase of the free oscillation are allowed to vary slowly with time. The phase-space analysis is applicable to the study of it. However, this method becomes practically complicated, since the analysis is compelled to resort to the graphical solution in a high-dimensional state space.

Table 4.5 The frequency components contained in the steady-state solutions

Type	ω is in the neighborhood of ($k = 0.5$, $n_2/n_1 = 1.0$)	Steady states				Notes
		(1) $r_{10} = 0$, $r_{20} = 0$	(2) $r_{10} \neq 0$, $r_{20} = 0$	(3) $r_{10} = 0$, $r_{20} \neq 0$	(4) $r_{10} \neq 0$, $r_{20} \neq 0$	
A12	$2\omega_1 - \omega_2 = 0.219n_1$	ω	——	ω, ω_2	$\omega, \omega_{10}, \omega_{20}$	$2\omega_{10} - \omega_{20} = \omega$
A 3	$\omega_1/3 = 0.272n_1$	——	$\omega, 3\omega$	——	$\omega, 3\omega, \omega_2$	
A 8	$(\omega_2 - \omega_1)/2 = 0.299n_1$	ω	——	——	$\omega, \omega_{10}, \omega_{20}$	$\omega_{20} - \omega_{10} = 2\omega$
A 4	$\omega_2/3 = 0.471n_1$	——	——	$\omega, 3\omega$	$\omega, 3\omega, \omega_1$	
A 1	$\omega_1 = 0.817n_1$	——	ω	——	ω, ω_2	
A 7	$(\omega_1 + \omega_2)/2 = 1.115n_1$	ω	——	——	$\omega, \omega_{10}, \omega_{20}$	$\omega_{10} + \omega_{20} = 2\omega$
A 2	$\omega_2 = 1.414n_1$	——	——	ω	ω, ω_1	
A10	$2\omega_2 - \omega_1 = 2.012n_1$	ω	ω, ω_1	——	$\omega, \omega_{10}, \omega_{20}$	$2\omega_{20} - \omega_{10} = \omega$
A 5	$3\omega_1 = 2.450n_1$	ω	$\omega, \omega/3$	ω, ω_2	$\omega, \omega/3, \omega_2$ *	
A11	$2\omega_1 + \omega_2 = 3.047n_1$	ω	——	ω, ω_2	$\omega, \omega_{10}, \omega_{20}$	$2\omega_{10} + \omega_{20} = \omega$
A 9	$\omega_1 + 2\omega_2 = 3.645n_1$	ω	ω, ω_1	——	$\omega, \omega_{10}, \omega_{20}$	$\omega_{10} + 2\omega_{20} = \omega$
A 6	$3\omega_2 = 4.243n_1$	ω	ω, ω_1	$\omega, \omega/3$	$\omega, \omega/3, \omega_1$ *	
A13	Non-resonant Case	ω	ω, ω_1	ω, ω_2	$\omega, \omega_1, \omega_2$ *	

Notes: The steady states marked by (*) are unstable.

The column marked by (——) means that the steady-state solution does not exist.

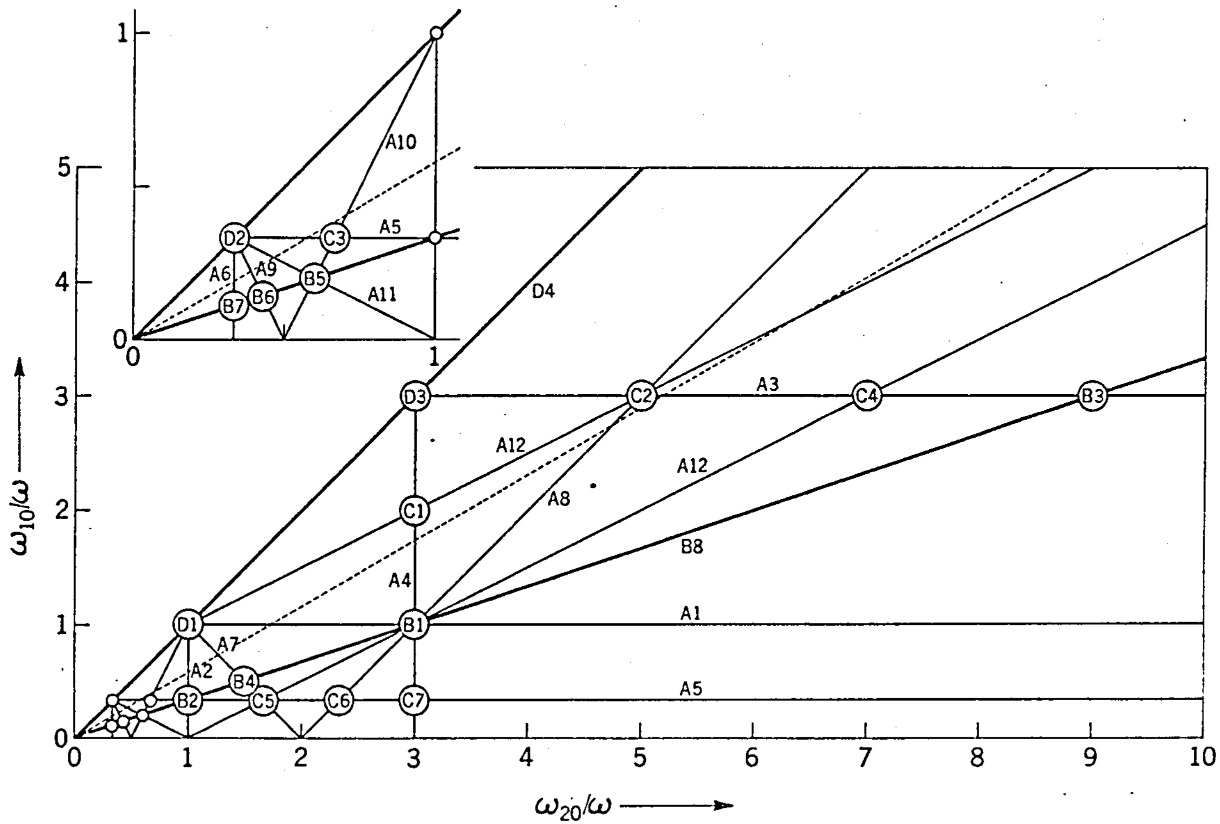


Fig. 4.1. Relationship among ω_{10}, ω_{20}, and ω when the external and internal resonances occur.

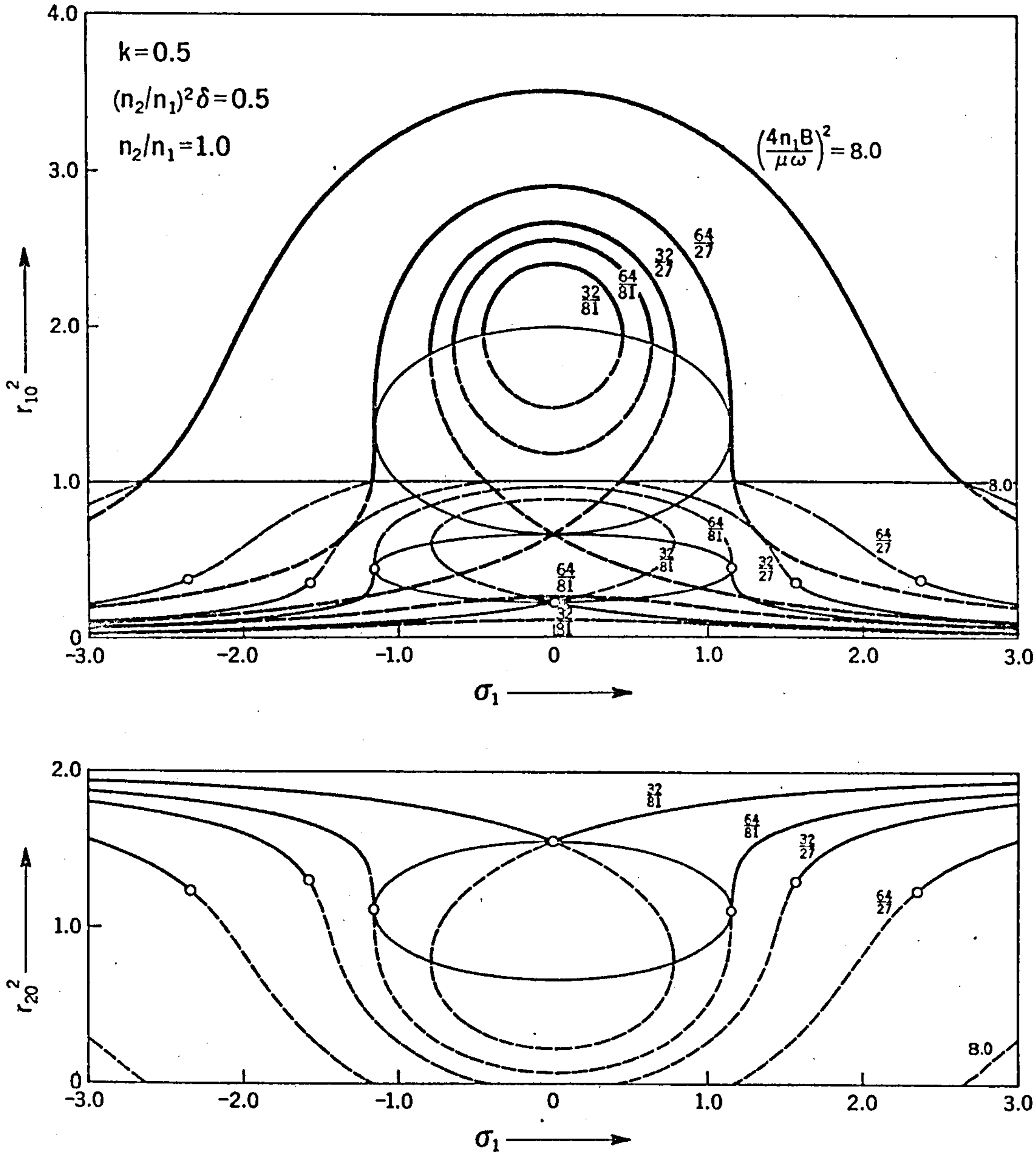


Fig. 4.2(a). Amplitude characteristics of the harmonic and combination oscillations.

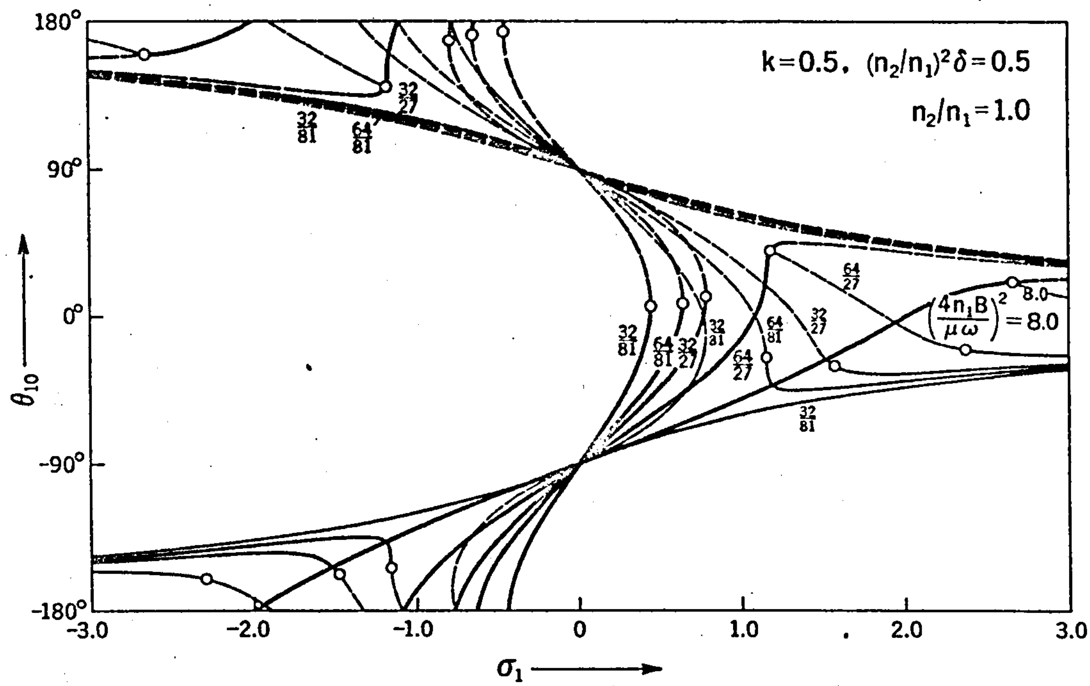


Fig. 4.2(b). Phase characteristics of the harmonic and combination oscillations.

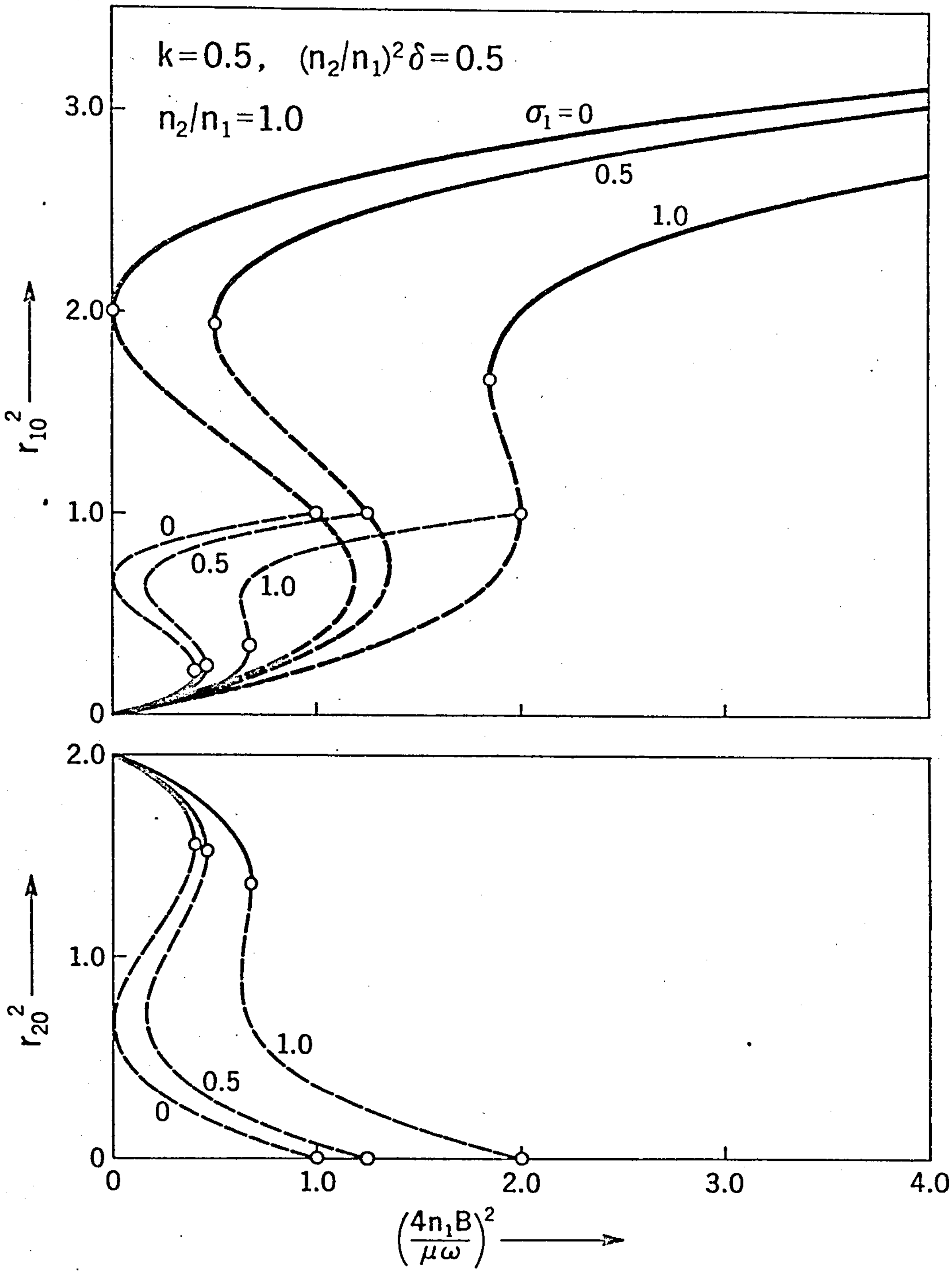


Fig. 4.3. Response curves with varying B ($\omega \simeq \omega_1$).

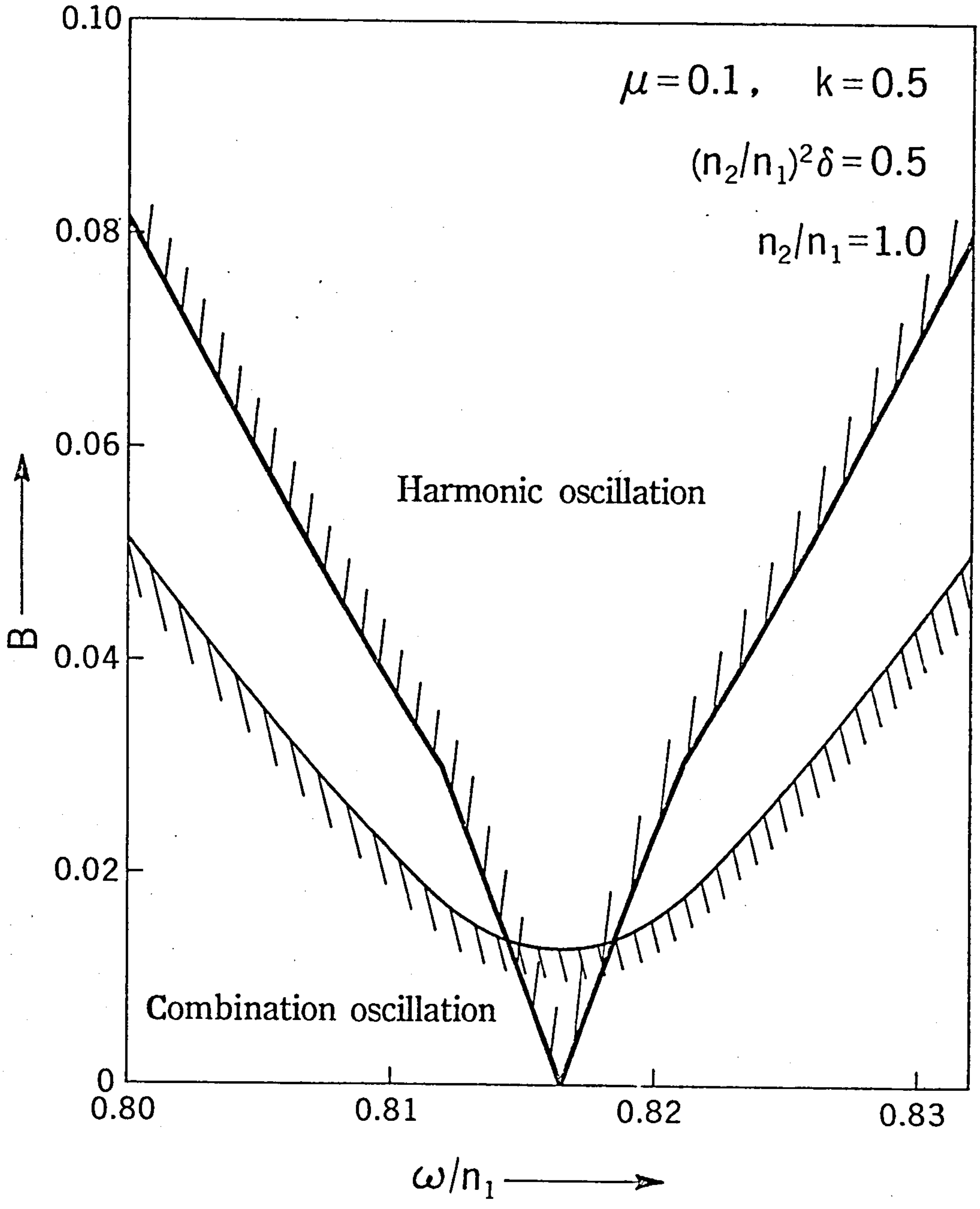


Fig. 4.4. Regions of frequency entrainment ($\omega \cong \omega_1$).

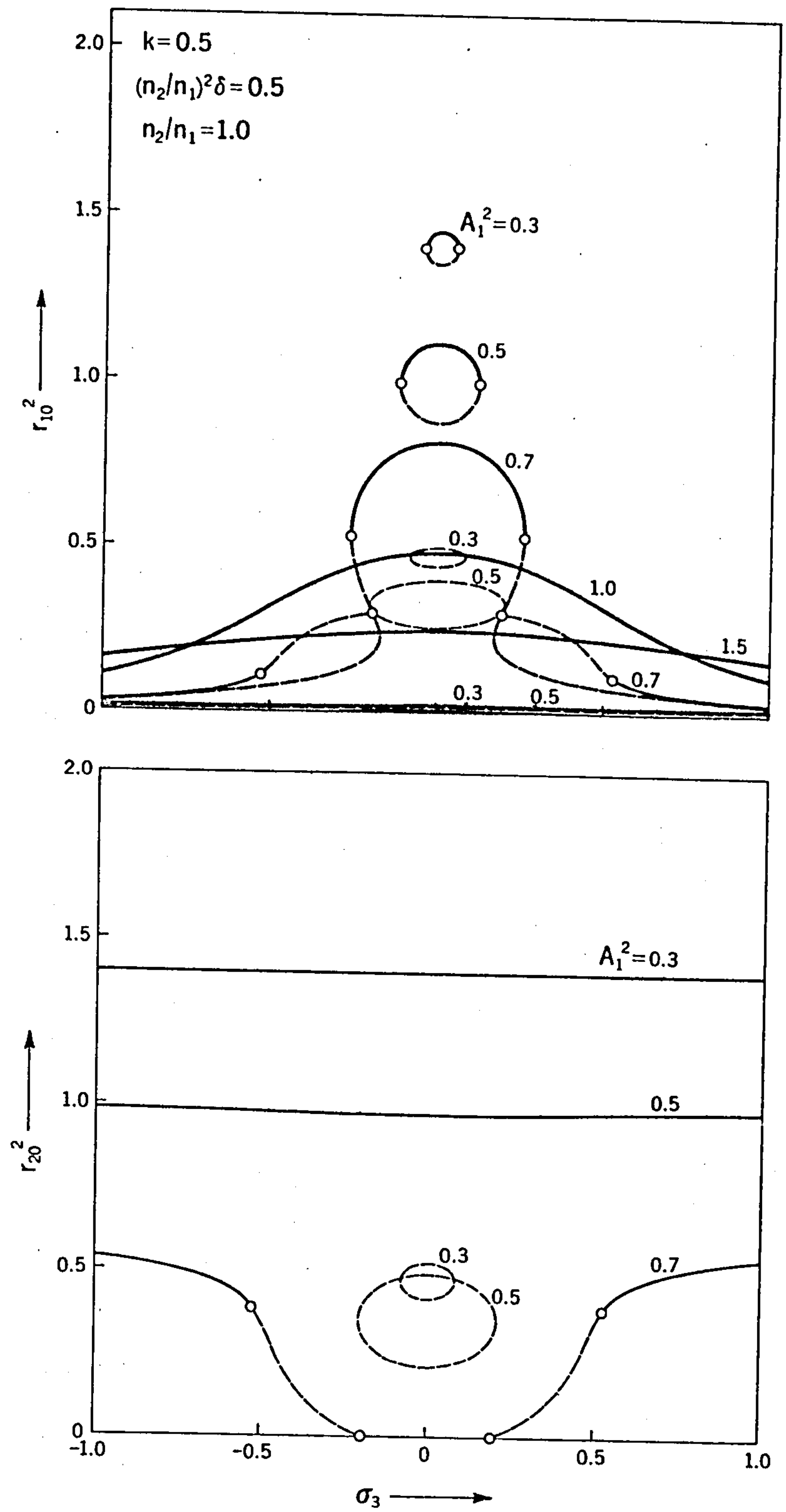


Fig. 4.5(a). Amplitude characteristics of the third harmonic and combination oscillations.

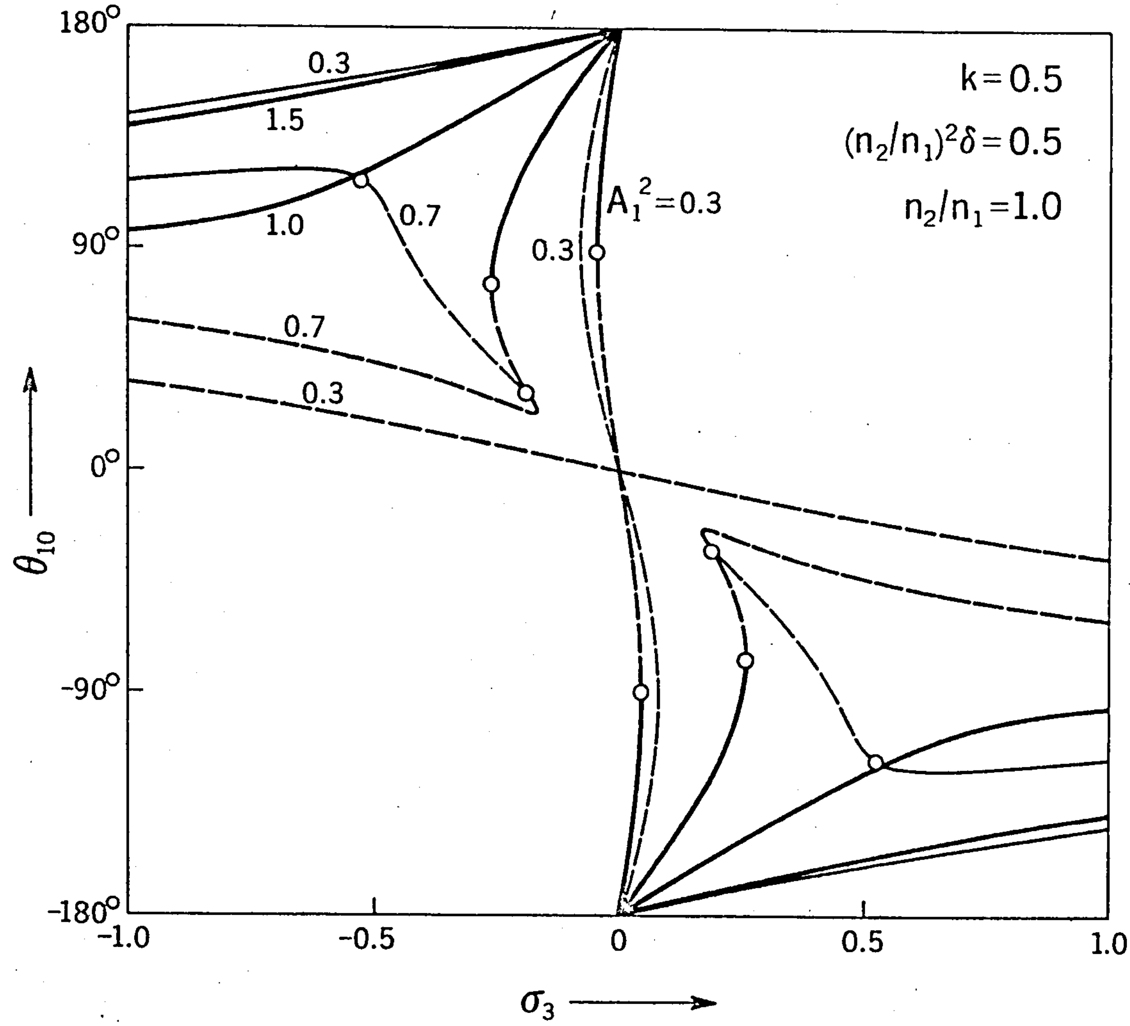


Fig. 4.5(b). Phase characteristics of the third-harmonic and combination oscillations.

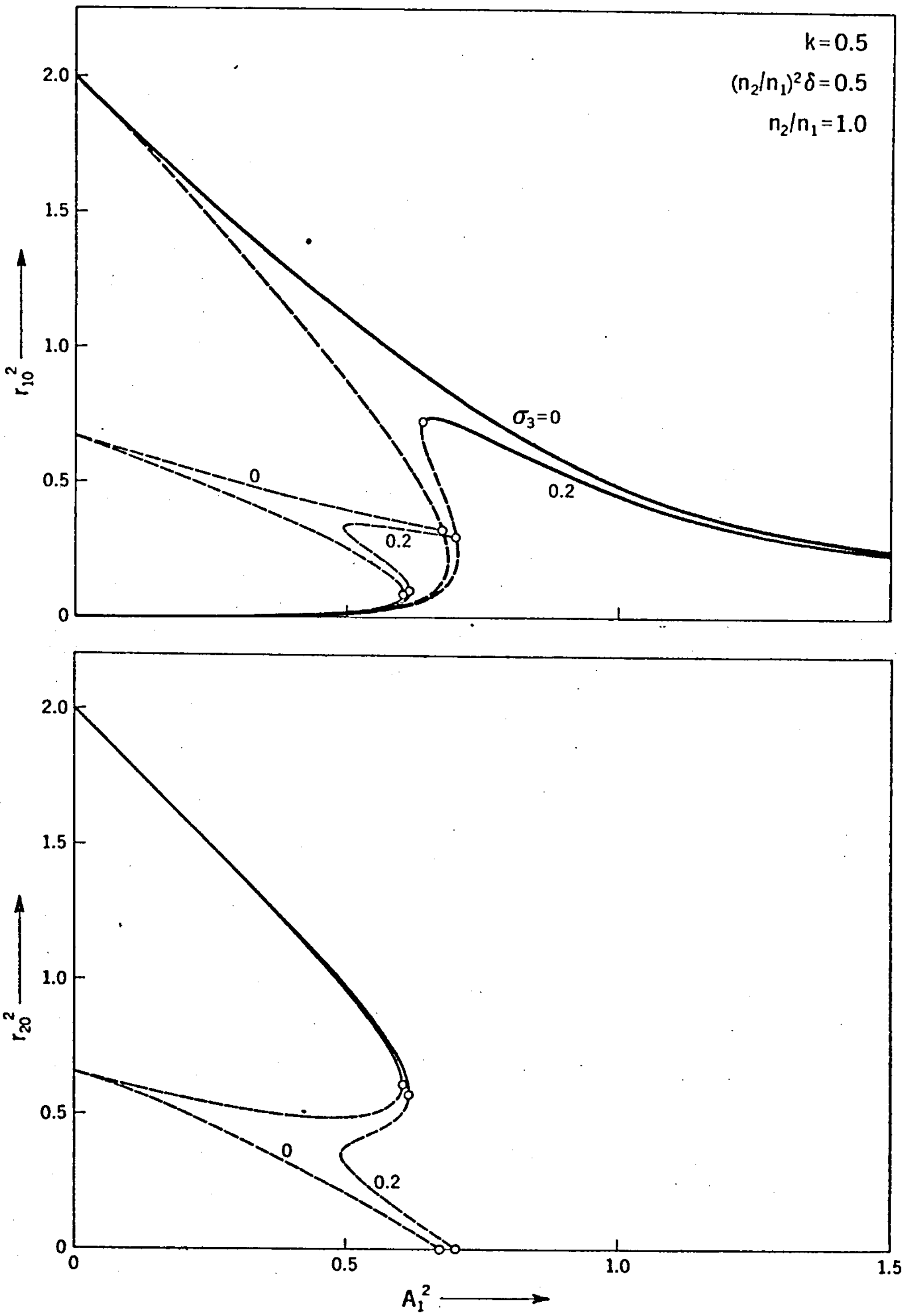


Fig. 4.6. Response curves with varying A_1 ($\omega \simeq \omega_1/3$).

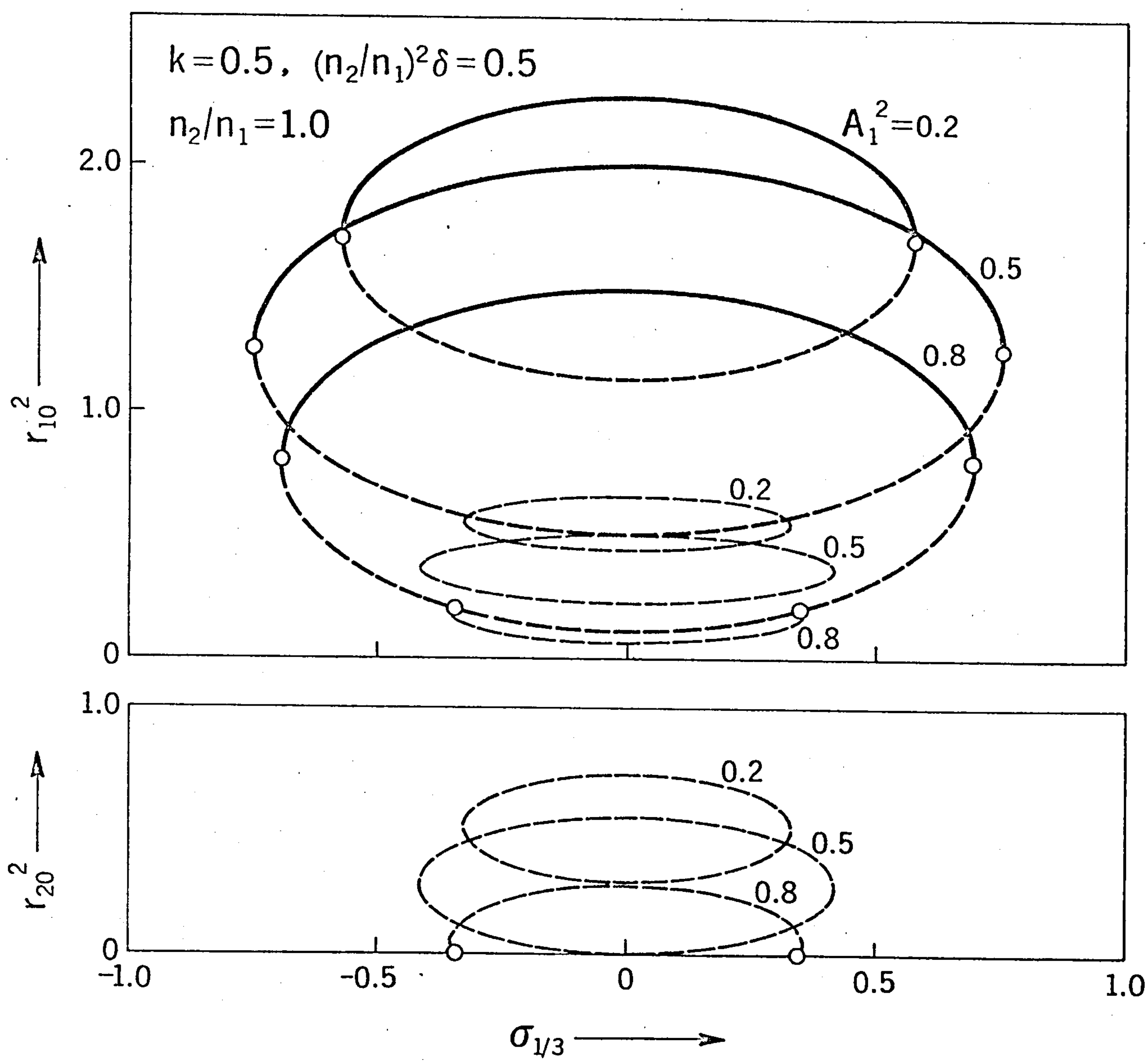


Fig. 4.7(a). Amplitude characteristics of the 1/3-harmonic and combination oscillations.

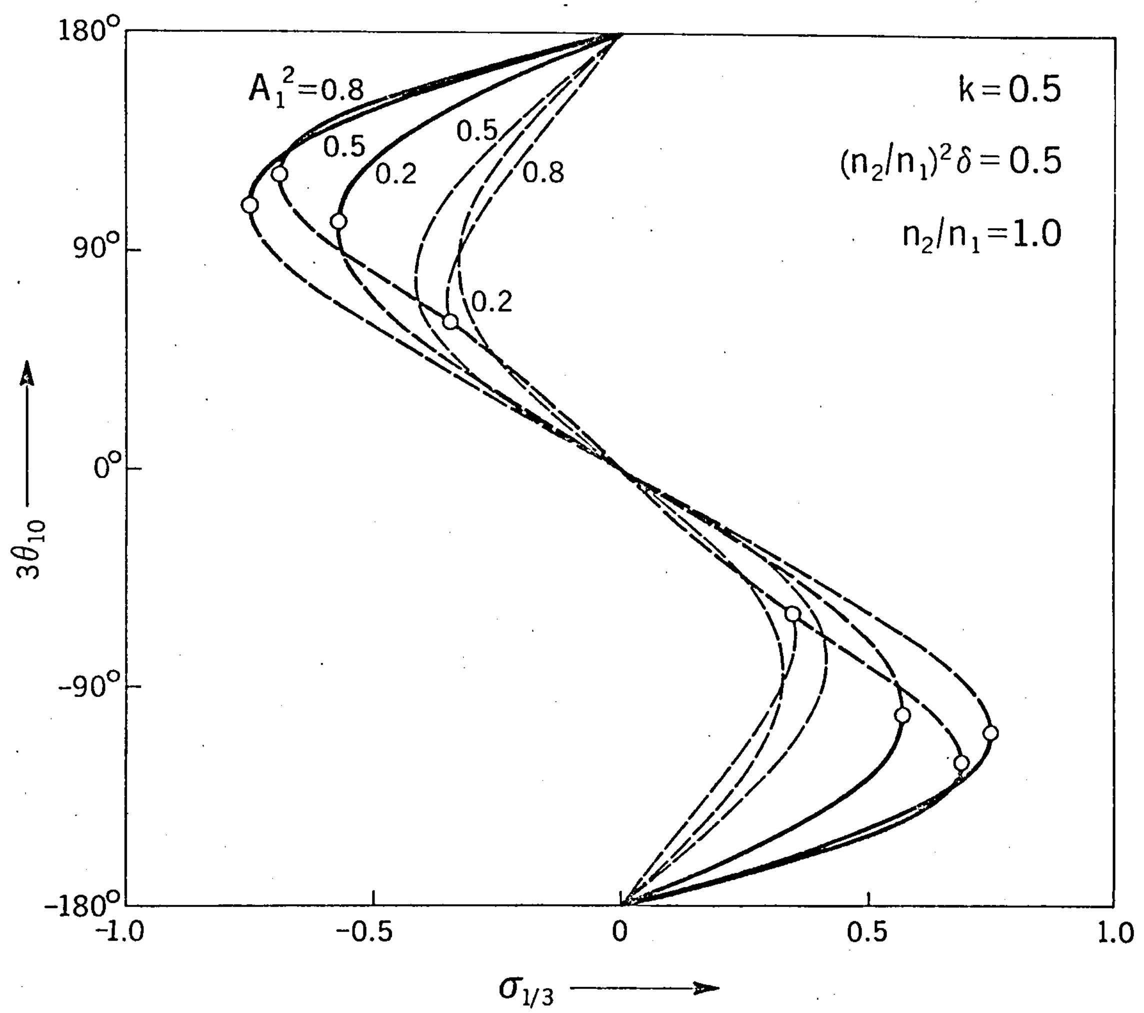


Fig. 4.7(b). Phase characteristics of the 1/3-harmonic and combination oscillations.

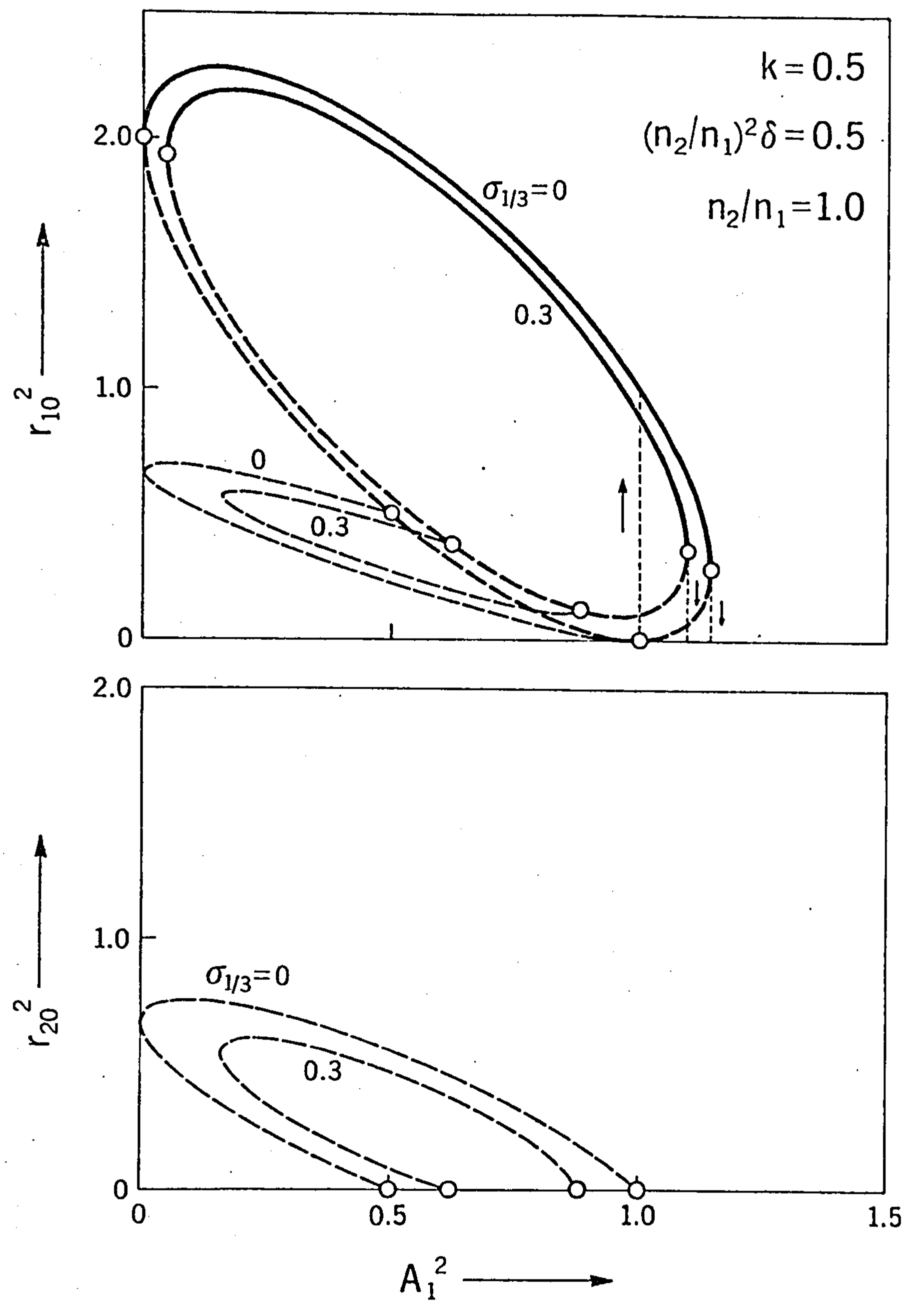


Fig. 4.8. Response curves with varying A_1 ($\omega \cong 3\omega_1$).

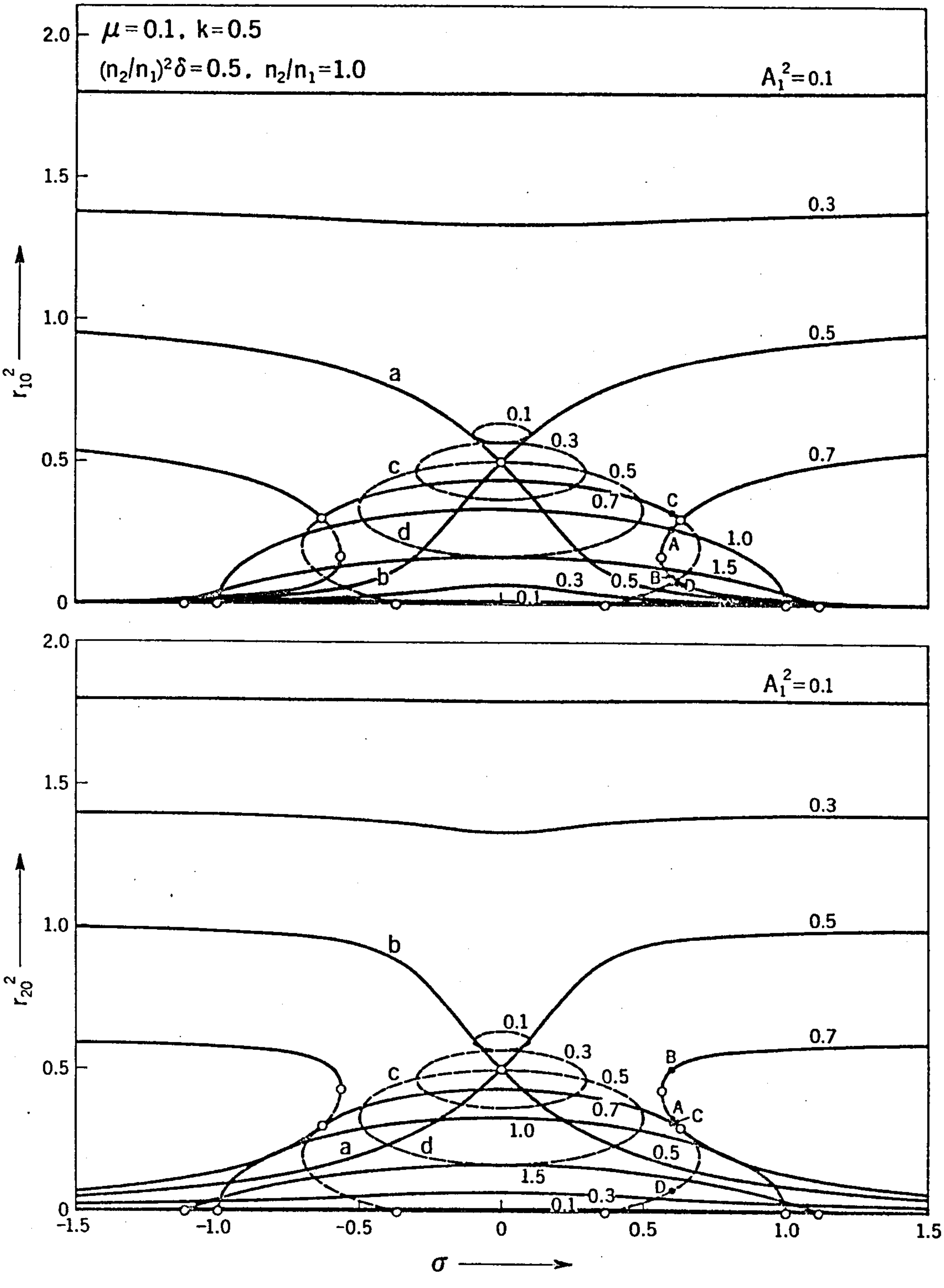


Fig. 4.9(a). Amplitude characteristic of the combination oscillation [$\omega \simeq (\omega_1 + \omega_2)/2$].

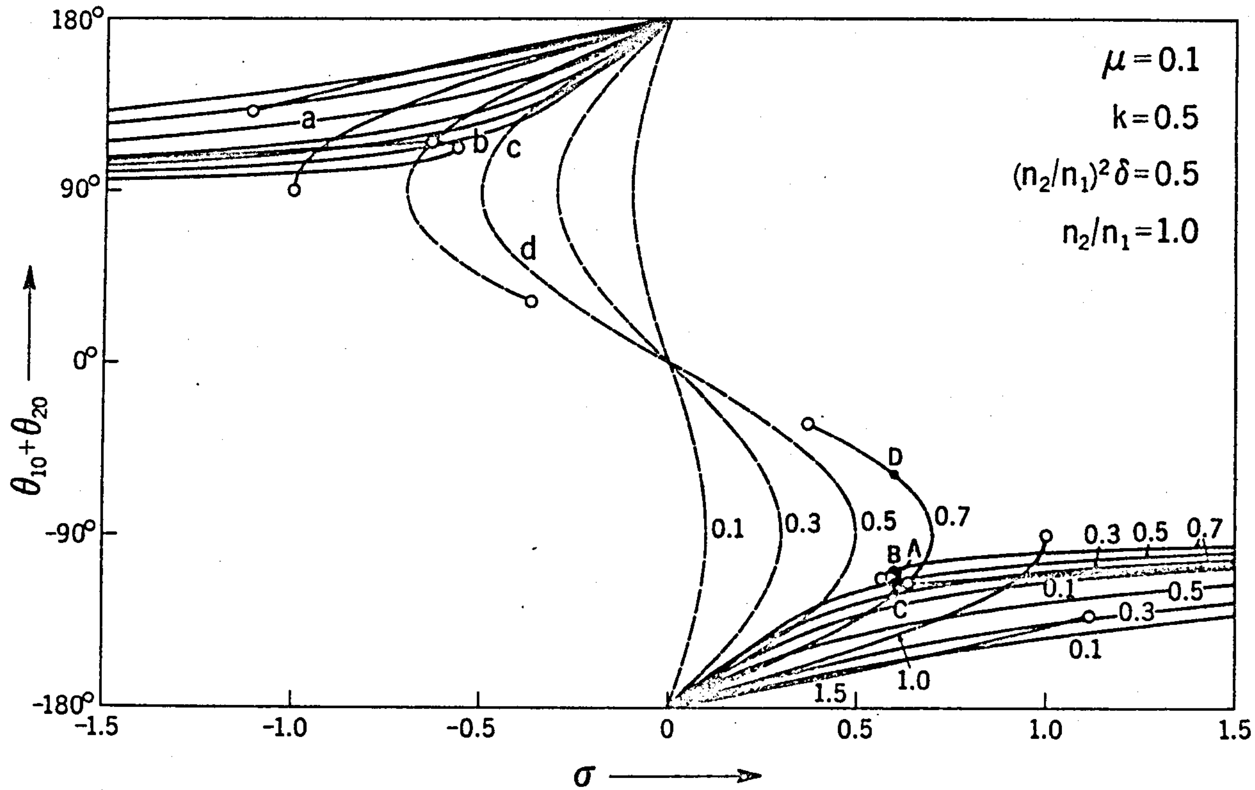


Fig. 4.9(b). Phase characteristic of the combination oscillation [$\omega \cong (\omega_1 + \omega_2)/2$].

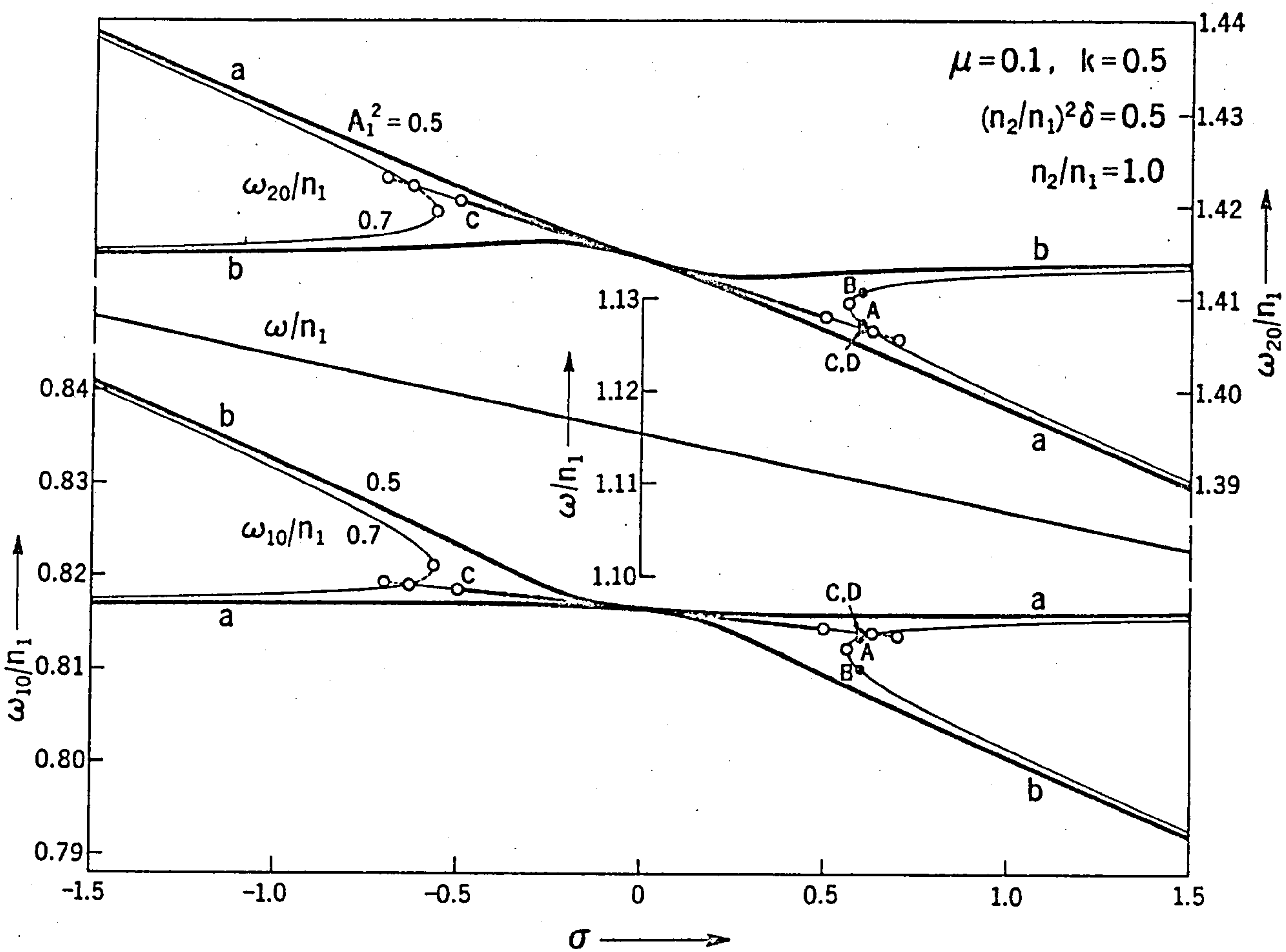
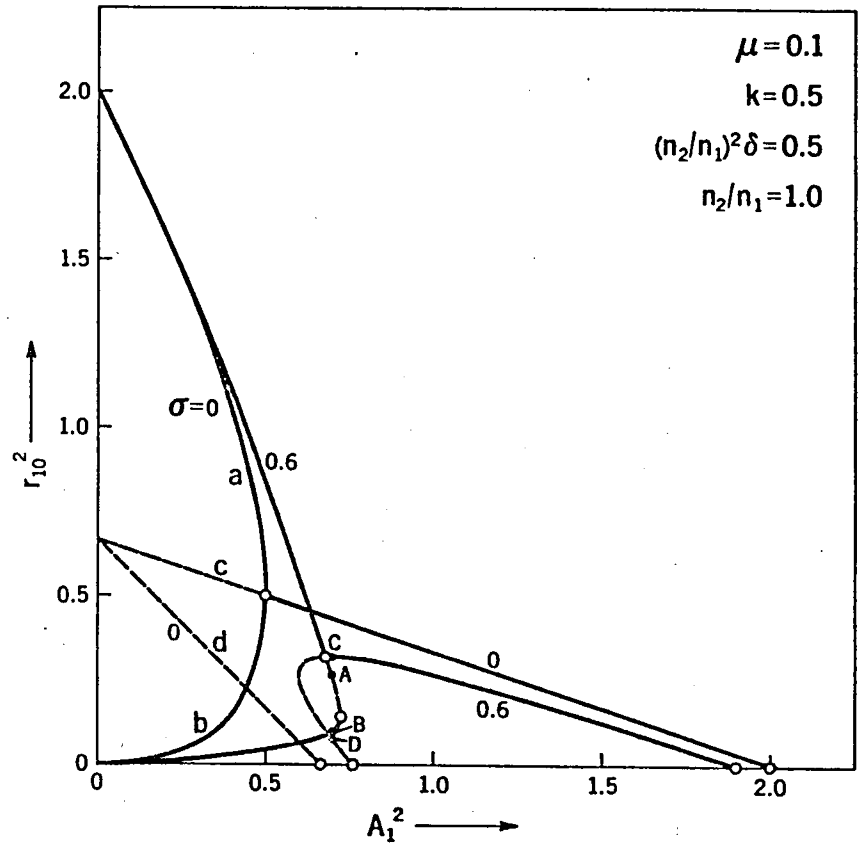


Fig. 4.9(c). Frequency characteristic of the combination oscillation [$\omega \cong (\omega_1 + \omega_2)/2$].



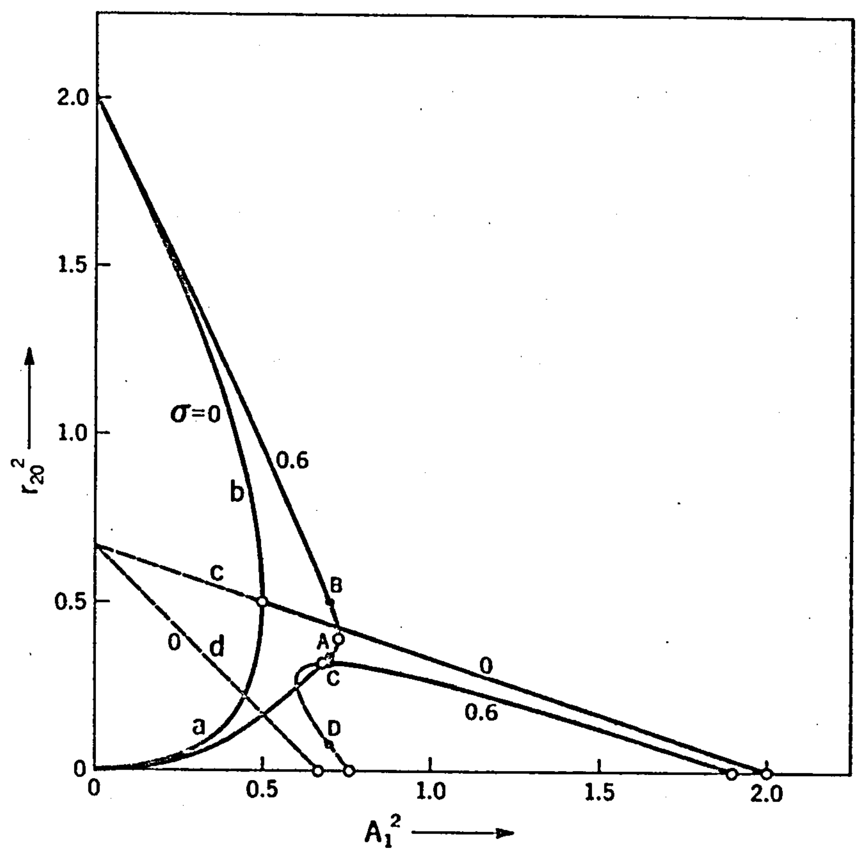


Fig. 4.10. Response curves with varying A_1 $[\omega \simeq (\omega_1 + \omega_2)/2]$.

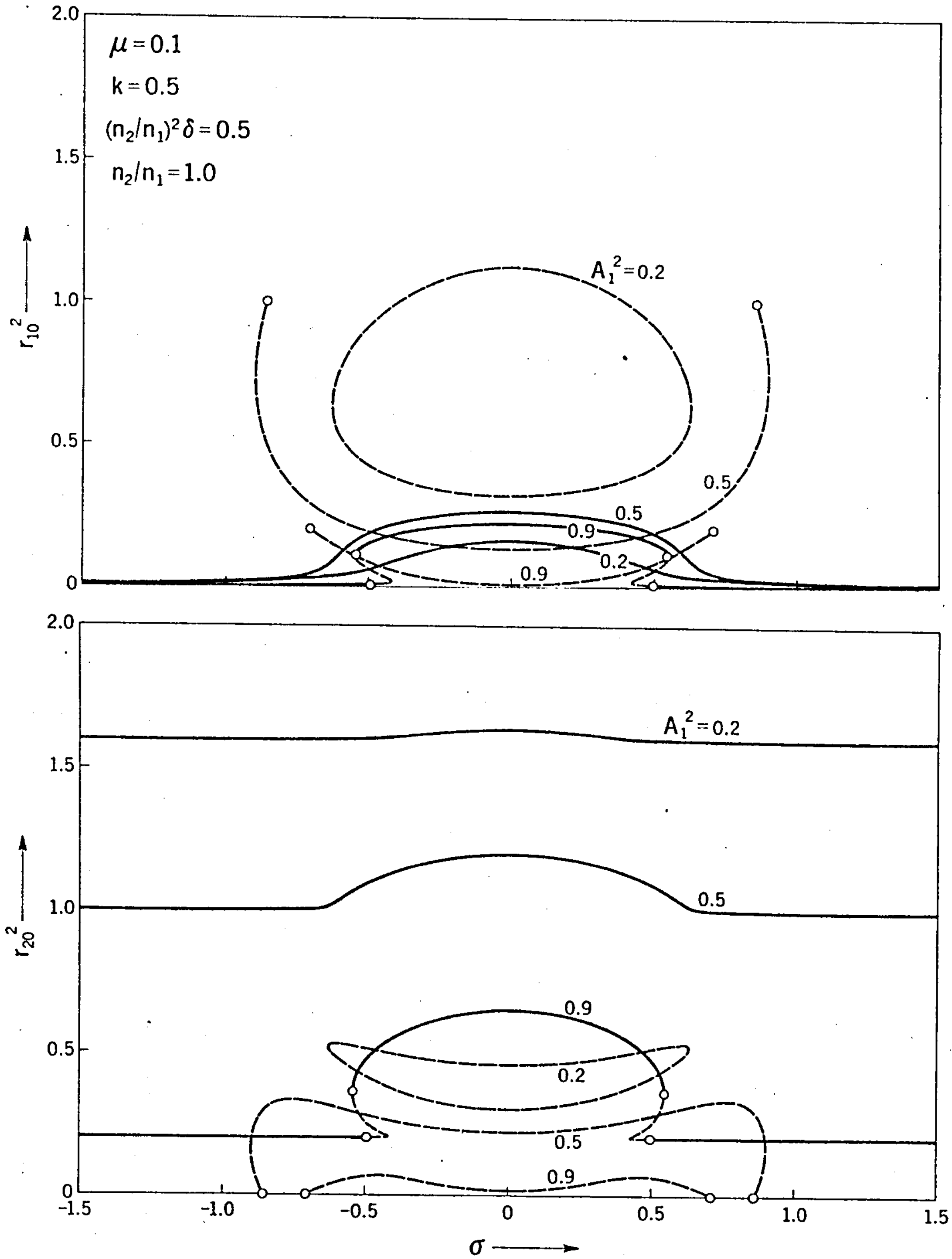


Fig. 4.11(a). Amplitude characteristic of the combination oscillation ($\omega \cong \omega_1 + 2\omega_2$).

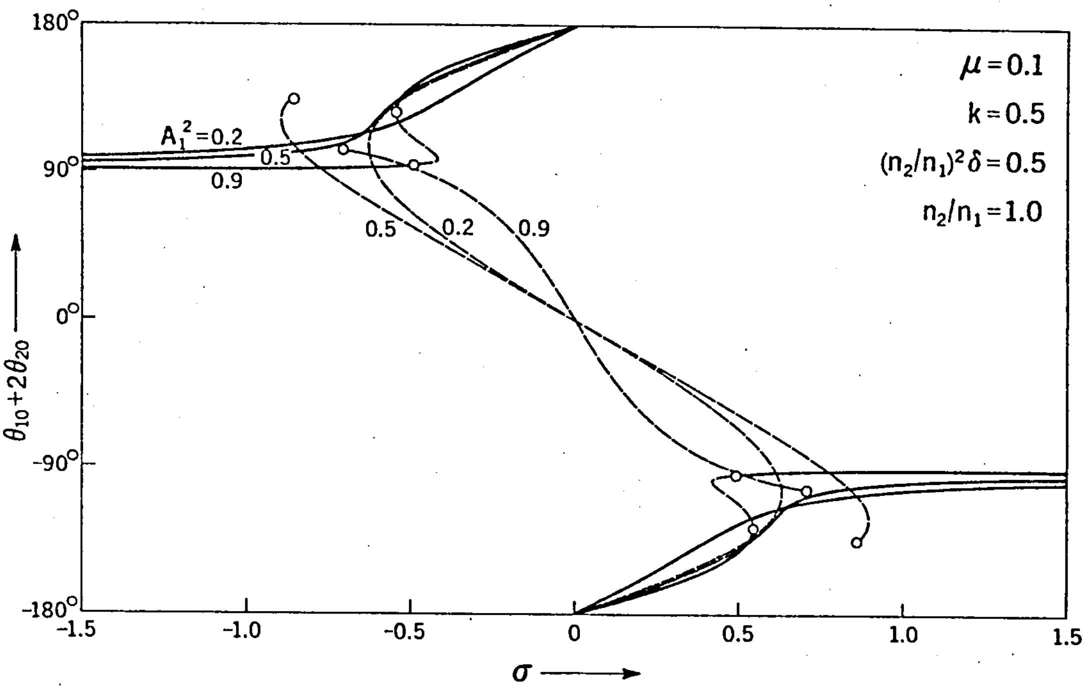


Fig. 4.11(b). Phase characteristic of the combination oscillation ($\omega \cong \omega_1 + 2\omega_2$).

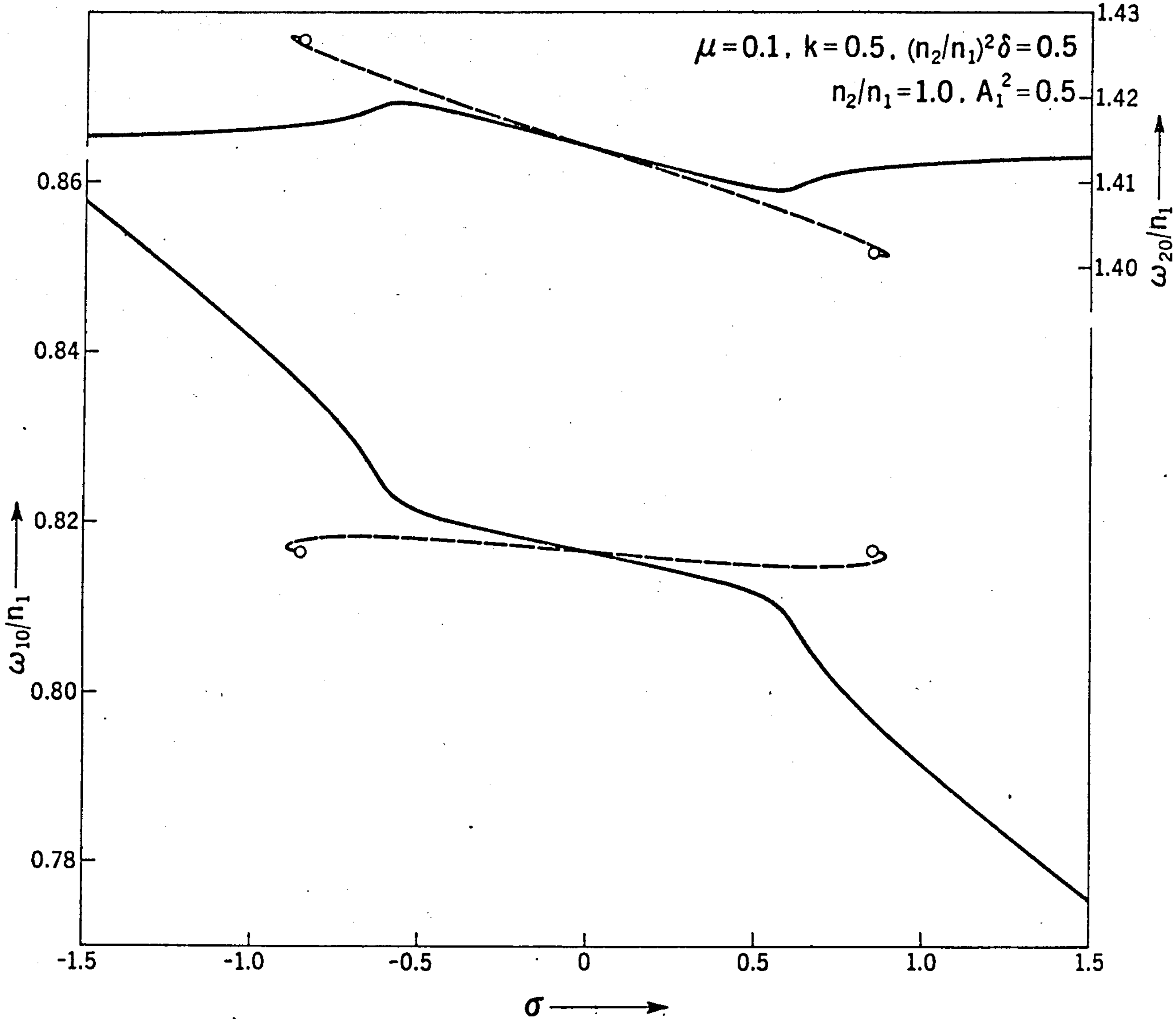


Fig. 4.11(c). Frequency characteristic of the combination oscillation ($\omega \cong \omega_1 + 2\omega_2$).

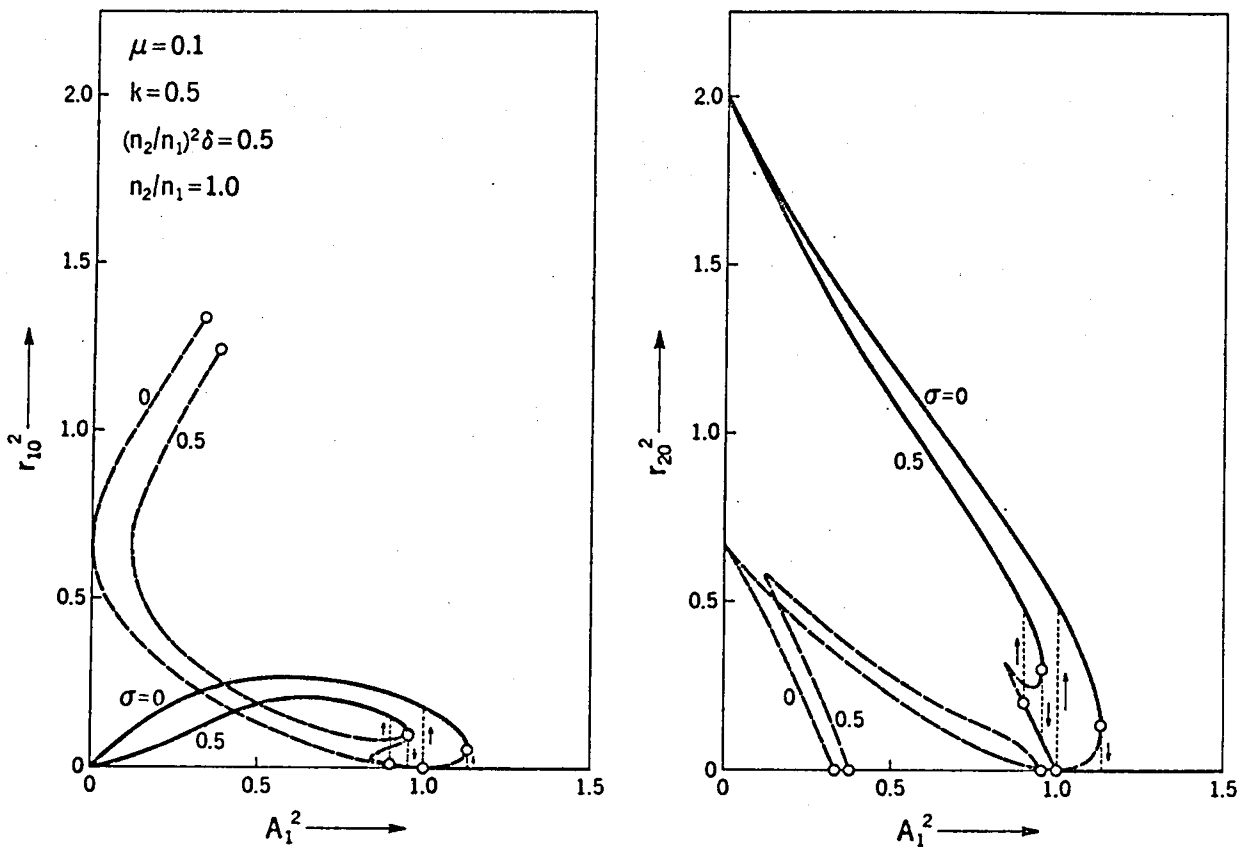


Fig. 4.12. Response curves with varying A_1 ($\omega \cong \omega_1 + 2\omega_2$).

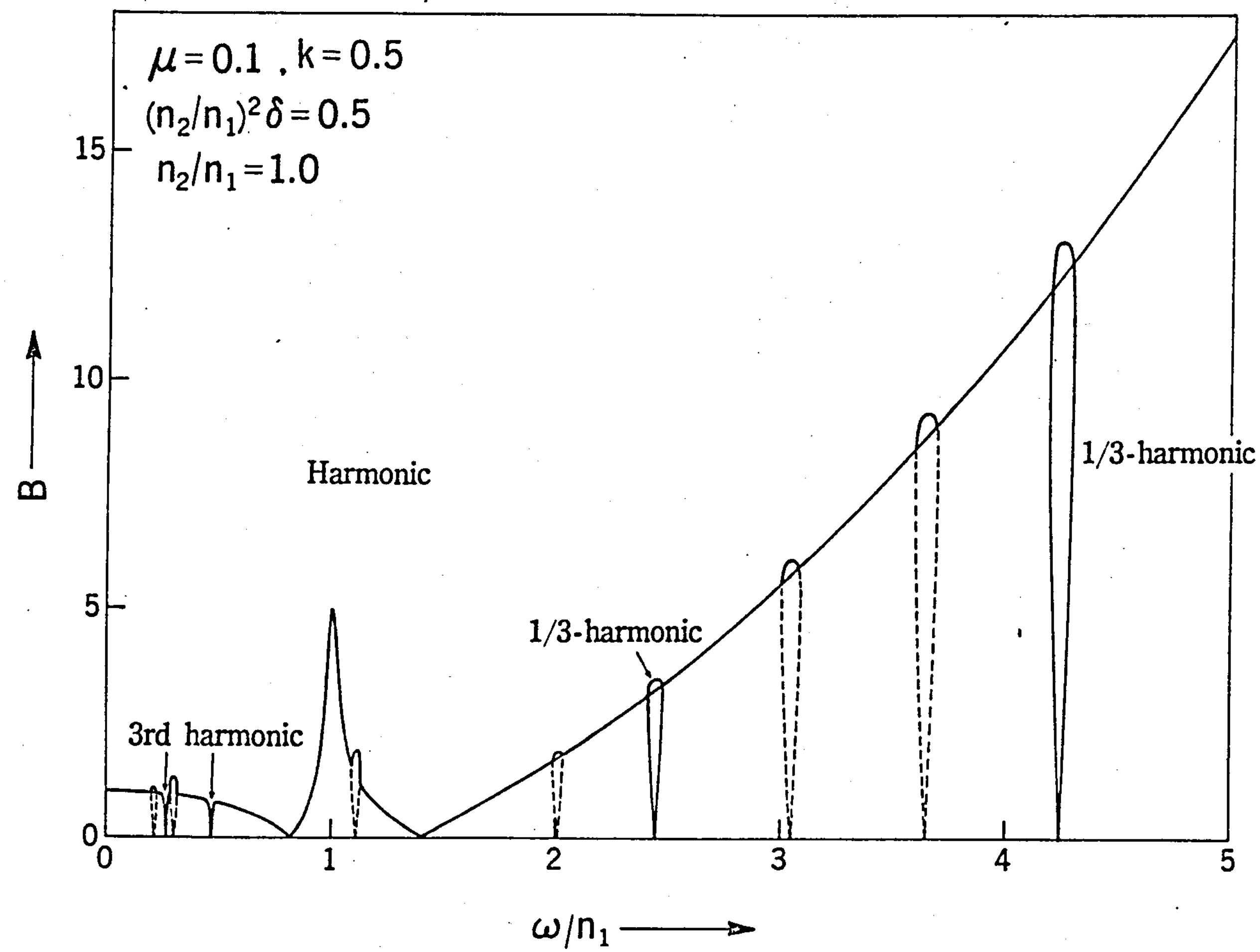


Fig. 4.13. Regions of frequency entrainment.

CHAPTER 5

INTERNAL RESONANCE UNDER LARGE COUPLING

BETWEEN TWO RESONANT CIRCUITS

5.1 Introduction

In the preceding chapter, we have investigated the behavior of a forced self-oscillatory system; however, we confined our discussion to the case where no internal resonance occurs. In this chapter, we treat the forced oscillations in a self-oscillatory system in which the internal resonance occurs.

As mentioned in Section 3.2, the internal resonance in the self-oscillatory system occurs if the ratio between the two natural frequencies is in the neighborhood of 1/3. When an external force is applied to this system, the external resonance may also occur. If one of the two natural frequencies is in the neighborhood of an integral multiple or submultiple of the driving frequency, so is the other natural frequency. Therefore, one may expect that both the two natural frequencies are entrained by the frequencies which are integral multiples or submultiples of the driving frequency. The resulting entrained oscillation becomes periodic with two or three components of frequencies. The amplitude characteristics of the entrained oscillations are obtained by using the averaging method. The stability of the oscillations is tested by making use of the Routh-Hurwitz criterion.

When the ratio between the two natural frequencies is in the neighborhood of an integer or a fraction (different from unity and 1/3), the entrainment of frequencies among the driving frequency and the two natural frequencies also occurs. This kind of entrainment is also discussed in this chapter.

When the coupling between the two resonant circuits of the oscillator is small and their resonant frequencies are close each other, the internal resonance also occurs. The forced oscillations in this system will be discussed in Chapter 6.

5.2 Derivation of Autonomous Systems by Using the Averaging Method

When the difference between the two natural frequencies of the system is large enough, the standard form of equations (1.21) and (1.25) were obtained in Chap. 1. By applying the averaging method for these equations, Eqs. (4.11) and (4.17) were derived in Chap. 4. Performing the integration in Eqs. (4.11) and (4.17) yields autonomous equations. The additional terms, however, appear in the right-hand sides of these equations, if there exists a certain relationship among ω_{10}, ω_{20}, and ω as listed in Tables 4.2 and 4.3. In this section we derive the autonomous systems for these cases.

(a) Autonomous Systems Derived When $3\omega_1 \cong \omega_2$ And $\omega \cong \omega_i$

We consider the cases where the external resonance, $\omega = \omega_{i0}$ ($i = 1, 2$), occurs besides the internal resonance, $3\omega_{10} = \omega_{20}$. Performing the integration in Eqs. (4.11) yields the following two autonomous systems.

B1:* When $\omega = \omega_{10} = \omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2) - \frac{4}{\omega}n_1 B_1 \sin\theta_1]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2)]$$

---

* The symbol corresponds to the classification of the type in Tables 4.2 and 4.3.

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[r_1^2 r_2 \sin(3\theta_1 - \theta_2) - \frac{4}{\omega}n_1 B_1 \cos\theta_1]\right\} \tag{5.1}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2} r_1^3 \sin(3\theta_1 - \theta_2)]$$

B2: When $\omega = 3\omega_{10} = \omega_{20}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[\rho_1 - r_1^2 - 2r_2^2 - r_1 r_2 \cos(3\theta_1 - \theta_2)]r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2) - \frac{4}{\omega}n_1 B_1 \sin\theta_2]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} r_1 r_2 \sin(3\theta_1 - \theta_2)]r_1 \tag{5.2}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\frac{1}{3}r_1^3 \sin(3\theta_1 - \theta_2) - \frac{4}{\omega}n_1 B_1 \cos\theta_2]\right\}$$

(b) Autonomous Systems Derived When $3\omega_1 \cong \omega_2$ And ω Is Not near ω_i

We treat the case where the internal resonance $3\omega_1 \cong \omega_2$ occurs but the driving frequency ω is not in the neighborhood of the natural frequencies ω_1 and ω_2. In this case we consider Eqs. (4.17). Performing the integration of Eqs. (4.17) yields the autonomous equations in what follows:

B3: When $\omega = \omega_{10}/3 = \omega_{20}/9$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2) - \frac{1}{3}A_1^3 \cos\theta_1]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[r_1^2 r_2 \sin(3\theta_1 - \theta_2) + \frac{1}{3}A_1^3 \sin\theta_1]\right\} \tag{5.3}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2} r_1^3 \sin(3\theta_1 - \theta_2)]$$

B4: When $\omega = 2\omega_{10} = 2\omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2) - A_1^2 r_2 \cos(\theta_1 + \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2) - A_1^2 r_1 \cos(\theta_1 + \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[r_1^2 r_2 \sin(3\theta_1 - \theta_2) + A_1^2 r_2 \sin(\theta_1 + \theta_2)]\right\} \tag{5.4}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\frac{1}{3}r_1^3 \sin(3\theta_1 - \theta_2) - A_1^2 r_1 \sin(\theta_1 + \theta_2)]\right\}$$

B5: When $\omega = 5\omega_{10} = 5\omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2) - A_1 r_2^2 \cos(\theta_1 - 2\theta_2) - 2A_1 r_1 r_2 \cos(2\theta_1 + \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2) - 2A_1 r_1 r_2 \cos(\theta_1 - 2\theta_2) - A_1 r_1^2 \cos(2\theta_1 + \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[r_1^2 r_2 \sin(3\theta_1 - \theta_2) + A_1 r_2^2 \sin(\theta_1 - 2\theta_2) + 2A_1 r_1 r_2 \sin(2\theta_1 + \theta_2)]\right\} \tag{5.5}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\frac{1}{3}r_1^3 \sin(3\theta_1 - \theta_2) + 2A_1 r_1 r_2 \sin(\theta_1 - 2\theta_2) - A_1 r_1^2 \sin(2\theta_1 + \theta_2)]\right\}$$

B6: When $\omega = 7\omega_{10} = 7\omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - r_1^2 r_2 \cos(3\theta_1 - \theta_2)$$
$$- A_1 r_2^2(\theta_1 + 2\theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2)$$
$$- 2A_1 r_1 r_2 \cos(\theta_1 + 2\theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[r_1^2 r_2 \sin(3\theta_1 - \theta_2) + A_1 r_2^2 \sin(\theta_1 + 2\theta_2)]\right\} \tag{5.6}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\frac{1}{3}r_1^3 \sin(3\theta_1 - \theta_2) - 2A_1 r_1 r_2 \sin(\theta_1 + 2\theta_2)]\right\}$$

B7: When $\omega = 9\omega_{10} = 3\omega_{20}$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2) - r_1 r_2 \cos(3\theta_1 - \theta_2)]r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2)$$
$$- A_1 r_2^2 \cos 3\theta_2]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} r_1^2 r_2 \sin(3\theta_1 - \theta_2)] \tag{5.7}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[-\frac{1}{3}r_1^3 \sin(3\theta_1 - \theta_2) + A_1 r_2^2 \sin 3\theta_2]\right\}$$

B8: When $3\omega_{10} = \omega_{20}$ and no external resonance occurs, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2 - r_1 r_2 \cos(3\theta_1 - \theta_2)]r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}r_1^3 \cos(3\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} r_1 r_2 \sin(3\theta_1 - \theta_2)]r_1 \tag{5.8}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21}r_2 - \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2} r_1^3 \sin(3\theta_1 - \theta_2)]$$

(c) Autonomous Systems Derived When $m\omega_1 \cong n\omega_2$ (m, n : positive integers)

Performing the integration in Eqs. (4.17) yields different types of autonomous equations as follows:

C1: When $\omega = \omega_{10}/2 = \omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2 - 2A_1 r_2 \cos(2\theta_1 - \theta_2)]r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}A_1^3 \cos\theta_2$$
$$- A_1 r_1^2 \cos(2\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu[\omega_{11}r_1 + \frac{\omega_1^2}{4n_1}\frac{k_2}{k_2 - k_1} A_1 r_1 r_2 \sin(2\theta_1 - \theta_2)] \tag{5.9}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[\frac{1}{3}A_1^3 \sin\theta_2 - A_1 r_1^2 \sin(2\theta_1 - \theta_2)]\right\}$$

C2: When $\omega = \omega_{10}/3 = \omega_{20}/5$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - \frac{1}{3}A_1^3 \cos\theta_1 - A_1^2 r_2 \cos(\theta_1 - \theta_2)$$
$$- 2A_1 r_1 r_2 \cos(2\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1^2 r_1 \cos(\theta_1 - \theta_2)$$
$$- A_1 r_1^2 \cos(2\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}\left[\frac{1}{3}A_1^3 \sin\theta_1 + A_1^2 r_2 \sin(\theta_1 - \theta_2)\right.\right.$$
$$\left.\left. + 2A_1 r_1 r_2 \sin(2\theta_1 - \theta_2)\right]\right\} \tag{5.10}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}\left[A_1^2 r_1 \sin(\theta_1 - \theta_2) + A_1 r_1^2 \sin(2\theta_1 - \theta_2)\right]\right\}$$

C3: When $\omega = 3\omega_{10} = 3\omega_{20}/2$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}\left[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_1^2 \cos 3\theta_1\right.$$
$$\left. - A_1 r_2^2 \cos(\theta_1 - 2\theta_2)\right]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}\left[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - 2A_1 r_1 r_2 \cos(\theta_1 - 2\theta_2)\right]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}\left[A_1 r_1^2 \sin 3\theta_1 + A_1 r_2^2 \sin(\theta_1 - 2\theta_2)\right]\right\} \tag{5.11}$$

$$r_2\dot{\theta}_2 = \mu\left[\omega_{21}r_2 - \frac{\omega_2^2}{4n_1}\frac{k_1}{k_1 - k_2}A_1 r_1 r_2 \sin(\theta_1 - 2\theta_2)\right]$$

C4: When $\omega = \omega_{10}/3 = \omega_{20}/7$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}\left[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - \frac{1}{3}A_1^3 \cos\theta_1\right.$$
$$\left. - 2A_1 r_1 r_2 \cos(2\theta_1 - \theta_2)\right]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}\left[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1 r_1^2 \cos(2\theta_1 - \theta_2)\right]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11}r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}\left[\frac{1}{3}A_1^3 \sin\theta_1 + 2A_1 r_1 r_2 \sin(2\theta_1 - \theta_2)\right]\right\} \tag{5.12}$$

$$r_2\dot{\theta}_2 = \mu\left[\omega_{21}r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}A_1 r_1^2 \sin(2\theta_1 - \theta_2)\right]$$

C5: When $\omega = 3\omega_{10} = 3\omega_{20}/5$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_1^2 \cos 3\theta_1 - A_1^2 r_2 \cos(\theta_1 + \theta_2) - 2A_1 r_1 r_2 \cos(2\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1^2 r_1 \cos(\theta_1 + \theta_2) - A_1 r_1^2 \cos(2\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11} r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[A_1 r_1^2 \sin 3\theta_1 + A_1^2 r_2 \sin(\theta_1 + \theta_2) + 2A_1 r_1 r_2 \sin(2\theta_1 - \theta_2)]\right\} \tag{5.13}$$

$$r_2\dot{\theta}_2 = \mu\left\{\omega_{21} r_2 + \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[A_1^2 r_1 \sin(\theta_1 + \theta_2) - A_1 r_1^2 \sin(2\theta_1 - \theta_2)]\right\}$$

C6: When $\omega = 3\omega_{10} = 3\omega_{20}/7$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)r_1 - A_1 r_1^2 \cos 3\theta_1 - A_1^2 r_2 \cos(\theta_1 - \theta_2)]$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - A_1^2 r_1 \cos(\theta_1 - \theta_2)]$$

$$r_1\dot{\theta}_1 = \mu\left\{\omega_{11} r_1 + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}[A_1 r_1^2 \sin 3\theta_1 + A_1^2 r_2 \sin(\theta_1 - \theta_2)]\right\} \tag{5.14}$$

$$r_2\dot{\theta}_2 = \mu[\omega_{21} r_2 - \frac{\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2} A_1^2 r_1 \sin(\theta_1 - \theta_2)]$$

C7: When $\omega = 3\omega_{10} = \omega_{20}/3$, we obtain

$$\dot{r}_1 = \frac{\mu\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2 - A_1 r_1 \cos 3\theta_1)r_1$$

$$\dot{r}_2 = \frac{\mu\omega_2^2}{8n_1}\frac{k_1}{k_1 - k_2}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)r_2 - \frac{1}{3}A_1^3 \cos\theta_2]$$

$$r_1\dot{\theta}_1 = \mu(\omega_{11} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1}A_1 r_1 \sin 3\theta_1)r_1 \qquad (5.15)$$

$$r_2\dot{\theta}_2 = \mu(\omega_{11}r_2 + \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2}A_1^3 \sin\theta_2)$$

5.3 Entrained Oscillations in a System with Internal Resonance $3\omega_1 \cong \omega_2$

In this section we consider the frequency entrainment which occurs in a system with internal resonance $3\omega_{10} = \omega_{20}$.

The steady-state solutions of Eqs. (4.11) and (4.17) and their stability are investigated in the same manner as we have done in the preceding chapters. Hereafter, r_{i0} and θ_{i0} ($i = 1, 2$) denote the steady-state values of r_i and θ_i, respectively.

(a) Entrained Oscillations Which Occur When $\omega \cong \omega_1 \cong \omega_2/3$

Let us consider the steady-state solutions of Eqs. (5.1) in which r_i and θ_i are constant. From Eqs. (5.1), we have

$$(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10} - r_{10}^2 r_{20}\cos(3\theta_{10} - \theta_{20}) - \frac{4n_1}{\omega}B_1 \sin\theta_{10} = 0$$

$$(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20} - \frac{1}{3}r_{10}^3 \cos(3\theta_{10} - \theta_{20}) = 0$$

$$\omega_{11}r_{10} + \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}[r_{10}^2 r_{20}\sin(3\theta_{10} - \theta_{20}) - \frac{4n_1}{\omega}B_1\cos\theta_{10}] = 0 \qquad (5.16)$$

$$\omega_{21}r_{20} - \frac{\omega_2^2 k_1}{24n_1(k_1 - k_2)}r_{10}^3 \sin(3\theta_{10} - \theta_{20}) = 0$$

From the second and the fourth members of Eqs. (5.16), we obtain

$$\cos(3\theta_{10} - \theta_{20}) = \frac{3r_{20}}{r_{10}^3}(\rho_2 - 2r_{10}^2 - r_{20}^2)$$

$$\sin(3\theta_{10} - \theta_{20}) = \frac{3\sigma_2 r_{20}}{r_{10}^3} \tag{5.17}$$

where

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - 3\omega) \ : \ \text{detuning}$$

Substituting Eqs. (5.17) into the first and the third members of Eqs. (5.16) yields

$$\sin\theta_{10} = \frac{\omega}{4n_1B_1r_{10}}[(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]$$

$$\cos\theta_{10} = \frac{\omega}{4n_1B_1r_{10}}(\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2) \tag{5.18}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - \omega) \ : \ \text{detuning}$$

Eliminating $3\theta_{10} - \theta_{20}$ and θ_{10} from Eqs. (5.17) and (5.18), we obtain,

$$[(\rho_2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2]r_{20}^2 = \frac{1}{9}r_{10}^6$$

$$[(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 + (\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)^2 = \left(\frac{4n_1B_1}{\omega}\right)^2 r_{10}^2 \tag{5.19}$$

Solving Eqs. (5.19) gives the amplitudes r_{10} and r_{20}. By substituting these values into Eqs. (5.17) and (5.18), we obtain the phase angles θ_{10} and θ_{20}.

From the form of the solution (4.7), and the relations $\omega_{10} = \omega$, $\omega_{20} = 3\omega$, we see that the entrained oscillation is periodic of the frequencies ω and 3ω.

Stability Investigation

The stability of the solutions is tested by the behavior of the small variations from the steady-state values. The variational equations (4.42) are sought from Eqs. (5.1). The stability conditions for this steady state are given by (4.46). The coefficients a_{ij} of the variational equations (4.42) are

$$
\begin{aligned}
a_{11} &= \mu m_1[\rho_1 - 3r_{10}^2 - 2r_{10}^2 - 2r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})] \\
a_{12} &= -\mu m_1[4r_{10}r_{20} + r_{10}^2\cos(3\theta_{10} - \theta_{20})] \\
a_{13} &= \mu m_1[3r_{10}^2r_{20}\sin(3\theta_{10} - \theta_{20}) - \frac{4}{\omega}n_1B_1\cos\theta_{10}] \\
a_{14} &= -\mu m_1r_{10}^2r_{20}\sin(3\theta_{10} - \theta_{20}) \\
a_{21} &= -\mu m_2[4r_{10}r_{20} + r_{10}^2\cos(3\theta_{10} - \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= -3a_{24} = \mu m_2r_{10}^3\sin(3\theta_{10} - \theta_{20}) \\
a_{31} &= \mu m_1[r_{20}\sin(3\theta_{10} - \theta_{20}) + \frac{4n_1B_1}{\omega r_{10}^2}\cos\theta_{10}] \\
a_{32} &= \mu m_1r_{10}\sin(3\theta_{10} - \theta_{20}) \\
a_{33} &= \mu m_1[3r_{10}r_{20}\cos(3\theta_{10} - \theta_{20}) + \frac{4n_1B_1}{\omega r_{10}}\sin\theta_{10}] \\
a_{34} &= -\mu m_1r_{10}r_{20}\cos(3\theta_{10} - \theta_{20}) \\
a_{41} &= -\mu m_2\frac{r_{10}^2}{r_{20}}\sin(3\theta_{10} - \theta_{20}) \\
a_{42} &= \frac{1}{3}\mu m_2\frac{r_{10}^3}{r_{20}^2}\sin(3\theta_{10} - \theta_{20}) \\
a_{43} &= -3a_{44} = -\mu m_2\frac{r_{10}^3}{r_{20}}\cos(3\theta_{10} - \theta_{20})
\end{aligned}
\tag{5.20}
$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

Numerical Example

The special cases we consider are those stemming from use of the following values* of the parameters, i.e.,

Case 1. $\mu = 0.1$ $k = 0.5$ $(n_2/n_1)^2\delta = 0.5$

Case 2. $\mu = 0.2$ $k = 0.8$ $(n_2/n_1)^2\delta = 0.5$

In Case 1, as mentioned in Sec. 3.2, we found that $3\omega_1 = \omega_2$, provided $n_2/n_1 = 0.403$ and $n_2/n_1 = 2.484$.† The amplitudes r_{10}, r_{20}, and the phase angles θ_{10}, θ_{20} are calculated by using Eqs. (5.17), (5.18), and (5.19). Figures 5.1 and 5.2 show the response curves of the entrained oscillations when $n_2/n_1 = 2.484$. The characteristic curve r_{10} of the harmonic component is similar to that of the harmonic entrainment [see Fig. 4.3], but the third harmonic component r_{20} also exists in this case. The stability of the solutions is tested by using the conditions (4.46), Eqs. (4.45), and (5.20). The unstable portions of the characteristic curves are shown by broken lines. In Fig. 5.3 the amplitudes r_{10} and r_{20} are illustrated for $n_2/n_1 = 0.403$ and $\omega/n_1 = 0.394$; ($\sigma_1 = \sigma_2 = 0$). In this case,

---

* The characteristics of the self-excited oscillations for these values were treated in Chaps. 2 and 3.

† It is to be noted that only the self-excited oscillation with the frequency ω_2 is stably sustained for $n_2/n_1 = 0.403$ because $\rho_1 < 0$ and $\rho_2 > 0$ [see Eqs. (2.8) and Fig. 2.2]. On the other hand, the entrained oscillation which has the third harmonic component predominantly occurs for $n_2/n_1 = 2.484$ because $\rho_1 > 0$ and $\rho_2 < 0$ [see Fig. 3.2].

since the self-excited oscillation having the third harmonic component predominantly does not occur in the self-oscillatory system, only the higher-harmonic entrainment between ω and ω_{20} occurs. We see that the amplitude characteristic r_{20} of the third harmonic component is similar to that shown in Fig. 4.6a and that the amplitude r_{10} is approximately proportional to B.

In Case 2, the relation $3\omega_1 = \omega_2$ is satisfied provided $n_2/n_1 = 1.0$.* When ω is equal to ω_1, the amplitude characteristics r_{10} and r_{20} are calculated and shown in Fig. 5.4. We see that there are nine different states of equilibrium, two of which (a and e) are stable under certain intervals of B. Figure 5.5 shows the response curves for B = 0.02, i.e., the relationships between ω and r_{10}, r_{20}, θ_{10}, θ_{20}, respectively.

The solutions of the fundamental equations (1.7) are obtained by using an analog computer. Typical waveforms of the entrained oscillation and their harmonic components are illustrated in Fig. 5.6.

(b) Entrained Oscillations Which Occur When $\omega \cong 3\omega_1 \cong \omega_2$

The steady-state solutions of Eqs. (5.2) are obtained by solving

$$[\rho_1 - r_{10}^2 - 2r_{20}^2 - r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})]r_{10} = 0$$

$$[(\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20} - \frac{1}{3}r_{10}^3\cos(3\theta_{10} - \theta_{20}) - \frac{4}{\omega}n_1B_1\sin\theta_{20}] = 0$$

---

* By using these values of the system parameters, we obtain $k_1 = -1$, $k_2 = 1$, and $\rho_1 = \rho_2 = 2$ [see Eqs. (1.13) and (2.8)]. Hence, in the self-oscillatory system (B = 0), two kinds of periodic oscillations are stably sustained: one is the harmonic oscillation with the frequency ω_2, and the other has the frequencies ω_1 and $3\omega_1$ (= ω_2) [see Fig. 3.3].

$$[\omega_{11} + \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)} r_{10} r_{20} \sin(3\theta_{10} - \theta_{20})] r_{10} = 0 \tag{5.21}$$

$$\omega_{21} r_{20} - \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)} [\frac{1}{3} r_{10}^3 \sin(3\theta_{10} - \theta_{20}) - \frac{4}{\omega} n_1 B_1 \cos\theta_{20}] = 0$$

We see, from Eqs. (5.21), that there are two types of steady states, i.e.,

$$\begin{aligned} &(1) \quad r_{10} = 0, \quad r_{20} \neq 0 \\ &(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0 \end{aligned} \tag{5.22}$$

From the form of the solution (4.7) and the relations $\omega_{10} = \omega/3$, $\omega_{20} = \omega$, we see that the steady state (1) corresponds to the periodic oscillation with the frequency ω, i.e., the harmonic entrainment. In the steady state (2), the natural frequency ω_1 is entrained by the driving frequency ω, and ω_2 is entrained by the frequency 1/3 times the driving frequency ω. Hence the resulting oscillation is periodic with the frequencies ω and $\omega/3$.

(1) Steady state : $r_{10} = 0$, $r_{20} \neq 0$

In the same manner as in the preceding paragraph, the amplitude r_{20} and the phase angle θ_{20} of the steady state (1) are obtained by solving

$$[(\rho_2 - r_{20})^2 + \sigma_2^2] r_{20}^2 = \left(\frac{4n_1 B_1}{\omega}\right)^2$$

and

$$\begin{aligned} \sin\theta_{20} &= \frac{\omega r_{20}}{4n_1 B_1}(\rho_2 - r_{20}^2) \\ \cos\theta_{20} &= -\frac{\omega\sigma_2 r_{20}}{4n_1 B_1} \end{aligned} \tag{5.23}$$

where

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - \omega) \ : \ \text{detuning}$$

This type of the solution is already discussed in Sec. 4.3.1.

(2) Steady state : $r_{10} \neq 0$, $r_{20} \neq 0$

We are particularly interested in the steady state (2). In the same manner as before the amplitudes r_{10} and r_{20} are obtained by solving

$$(\rho_1 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_1^2 = r_{10}^2 r_{20}^2$$

$$\begin{aligned}&[\tfrac{1}{3}(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\ &\quad + (\tfrac{\sigma_1}{3} r_{10}^2 + \sigma_2 r_{20}^2)^2 = \left(\frac{4n_1 B_1}{\omega}\right)^2 r_{20}^2\end{aligned} \tag{5.24}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - \omega/3) \text{ : detuning}$$

and σ_2 is given by Eqs. (5.23). The phase angles θ_{10} and θ_{20} are given by

$$\begin{aligned}\sin\theta_{20} &= \frac{\omega}{4n_1 B_1 r_{20}}[-\tfrac{1}{3}(\rho_1 - r_{10}^2 - 2r_{20}^2)r_{10}^2 + (\rho_2 - 2r_{10}^2 - r_{20}^2)r_{20}^2] \\ \cos\theta_{20} &= -\frac{\omega}{4n_1 B_1 r_{20}}(\tfrac{1}{3}\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2) \\ \sin(3\theta_{10} - \theta_{20}) &= -\frac{\sigma_1}{r_{10} r_{20}} \\ \cos(3\theta_{10} - \theta_{20}) &= \frac{1}{r_{10} r_{20}}(\rho_1 - r_{10}^2 - 2r_{20}^2)\end{aligned} \tag{5.25}$$

Stability Investigation

The stability of the steady-state solutions is tested as before. The variational equations (4.42) are sought from Eqs. (5.2). Their coefficients a_{ij} are given by

$$a_{11} = \mu m_1[\rho_1 - 3r_{10}^2 - 2r_{20}^2 - 2r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})]$$

$$a_{12} = -\mu m_1[4r_{10}r_{20} + r_{10}^2\cos(3\theta_{10} - \theta_{20})]$$

$$a_{13} = -3a_{14} = 3\mu m_1 r_{10}^2 r_{20}\sin(3\theta_{10} - \theta_{20})$$

$$
\begin{aligned}
a_{21} &= -\mu m_2[4r_{10}r_{20} + r_{10}^2 \cos(3\theta_{10} - \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= \mu m_2 r_{10}^3 \sin(3\theta_{10} - \theta_{20}) \\
a_{24} &= \mu m_2[-\tfrac{1}{3} r_{10}^3 \sin(3\theta_{10} - \theta_{20}) - \tfrac{4}{\omega} n_1 B_1 \cos\theta_{20}] \\
a_{31} &= \mu m_1 r_{20} \sin(3\theta_{10} - \theta_{20}) \\
a_{32} &= \mu m_1 r_{10} \sin(3\theta_{10} - \theta_{20}) \\
a_{33} &= -3a_{34} = 3\mu m_1 r_{10} r_{20} \cos(3\theta_{10} - \theta_{20}) \\
a_{41} &= -\mu m_2 \frac{r_{10}^2}{r_{20}} \sin(3\theta_{10} - \theta_{20}) \\
a_{42} &= \mu m_2[\frac{1}{3}\frac{r_{10}^3}{r_{20}^2} \sin(3\theta_{10} - \theta_{20}) + \frac{4n_1B_1}{\omega r_{20}^2}\cos\theta_{20}] \\
a_{43} &= -\mu m_2 \frac{r_{10}^3}{r_{20}} \cos(3\theta_{10} - \theta_{20}) \\
a_{44} &= \mu m_2[\frac{1}{3}\frac{r_{10}^3}{r_{20}} \cos(3\theta_{10} - \theta_{20}) + \frac{4n_1B_1}{\omega r_{20}}\sin\theta_{20}]
\end{aligned}
\tag{5.26}
$$

where

$$
m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}
$$

From Eqs. (5.26), we see that one of the characteristic roots λ for the steady state (1) is zero, because $a_{i3} = 0$ ($i = 1, .., 4$). Therefore, the stability conditions are given by (4.48). For the steady state (2), the stability conditions are given by (4.46).

<u>Numerical Examples</u>

Let us consider the cases in which the parameters of Eqs. (1.24) take the same values as in the preceding section, i.e.,

Case 1. $\mu = 0.1$ $k = 0.5$ $(n_2/n_1)^2\delta = 0.5$ $n_2/n_1 = 2.484$*

Case 2. $\mu = 0.2$ $k = 0.8$ $(n_2/n_1)^2\delta = 0.5$ $n_2/n_1 = 1.0$

The response characteristics calculated by using Eqs. (5.23) to (5.25) are illustrated in Figs. 5.7, 5.8, 5.9, and 5.10. In these figures, the thick lines show the oscillation with the subharmonic component. The characteristics of the harmonic entrainment are shown by fine lines. The unstable portions of the characteristic curves are shown by broken lines. We see that both the steady states (1) and (2) of Eqs. (5.22) are stable sustained.† The hysteresis between these two steady states is observed as the amplitude B of the driving frequency varies (Fig. 5.7). Typical waveforms of the entrained oscillations obtained by using an analog computer and their harmonic components are illustrated in Fig. 5.11.

(c) Combination Oscillations without External Resonance

We treat the case where the internal resonance $3\omega_{10} = \omega_{20}$ exists but any kind of external resonance does not occur.

---

* As mentioned in Sec. 3.2, the relations $3\omega_1 = \omega_2$ is also satisfied when the value of n_2/n_1 is decreased to 0.403 in Case 1. In this case, however, the entrained oscillation with the subharmonic component does not occur because $\rho_1 < 0$ in Eqs. (5.24). The characteristic of the harmonic entrainment is identical with that which is obtained when no internal resonance occurs (see Sec. 4.3.1).

† It depends on the initial condition as regards which kind of oscillations occurs.

The steady-state solutions of Eqs. (5.8) are obtained by solving

$$[\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2 - r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})]r_{10} = 0$$

$$[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} - \frac{1}{3} r_{10}^3 \cos(3\theta_{10} - \theta_{20})] = 0$$

$$\omega_{11}r_{10} + \frac{\omega_1^2}{8n_1}\frac{k_2}{k_2 - k_1} r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) = 0 \tag{5.27}$$

$$\omega_{21}r_{20} - \frac{\omega_2^2}{24n_1}\frac{k_1}{k_1 - k_2} r_{10}^3 \sin(3\theta_{10} - \theta_{20}) = 0$$

We see, from Eqs. (5.27), that there are three types of steady states, i.e.,

$$\begin{aligned} &(1)\quad r_{10} = 0,\quad r_{20} = 0 \\ &(2)\quad r_{10} = 0,\quad r_{20}^2 = \rho_2 - 2A_1^2,\quad \omega_{21} = 0 \\ &(3)\quad r_{10} \neq 0,\quad r_{20} \neq 0 \end{aligned} \tag{5.28}$$

From the form of the solution (4.16), we see that the steady-state solution is periodic of frequency ω in (1). In (2), since $\omega_{20} = \omega_2$, there exists a combination oscillation with two frequencies ω and ω_2. These two cases are identical with the first and the third cases of Eqs. (4.54). In the steady state (3), the solution is a combination oscillation having three frequency components ω, ω_{10}, and ω_{20} (= $3\omega_{10}$). Since ω and ω_{10} (= $\omega_{20}/3$) are generally incommensurable, this solution is almost periodic.

Eliminating $3\theta_{10} - \theta_{20}$ in Eqs. (5.27) as before yields

$$(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - 2r_{20}^2)r_{20}^2 = 0$$

$$\Big\{[3m_1(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) + m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)]^2 + (3\omega_{11} - \omega_{21})^2\Big\}r_{20}^2 - [3m_1r_{20}^2 + \frac{m_2}{3} r_{10}^2]^2 r_{10}^2 = 0 \tag{5.29}$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

From Eqs. (4.1), we see

$$\mu(3\omega_{11} - \omega_{21}) = 3\omega_1 - \omega_2 + O(\mu^2) \tag{5.30}$$

Solving Eqs. (5.29) simultaneously gives the amplitudes r_{10} and r_{20}. Then, the phase difference $3\theta_{10} - \theta_{20}$, and the frequencies ω_{11}, ω_{21} are sought from Eqs. (5.27), namely,

$$\begin{aligned}
\sin(3\theta_{10} - \theta_{20}) &= -\frac{(3\omega_{11} - \omega_{21})r_{20}}{(3m_1 r_{20}^2 + \frac{m_2}{3} r_{10}^2)r_{10}} \\
\cos(3\theta_{10} - \theta_{20}) &= \frac{1}{r_{10}r_{20}}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) \\
&= \frac{3r_{20}}{r_{10}^3}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2) \\
\omega_{11} &= -m_1 r_{10} r_{20} \sin(3\theta_{10} - \theta_{20}) \\
\omega_{21} &= \frac{m_2}{3}\frac{r_{10}^3}{r_{20}} \sin(3\theta_{10} - \theta_{20})
\end{aligned} \tag{5.31}$$

Hence, from Eqs. (4.1) and (5.31), the frequencies ω_{10} and ω_{20} are determined by

$$\begin{aligned}
\omega_{10} &= \omega_1 - \mu\omega_{11} \\
\omega_{20} &= \omega_2 - \mu\omega_{21}
\end{aligned} \tag{5.32}$$

Stability Investigation

The stability of the steady states (5.28) is studied by solving the variational equations derived from Eqs. (5.8). Since the first and the second cases of Eqs. (5.28) are identical with the first and the third cases of Eqs. (4.54), the stability conditions are also given by (4.56) and (4.58), respectively. For the third case of Eqs. (5.28), the coefficients of the variational equations

(4.42) are given by

$$
\begin{aligned}
a_{11} &= \mu m_1[\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2 - 2r_{10}r_{20}\cos(3\theta_{10} - \theta_{20})] \\
a_{12} &= -\mu m_1[4r_{10}r_{20} + r_{10}^2\cos(3\theta_{10} - \theta_{20})] \\
a_{13} &= -3a_{14} = 3\mu m_1 r_{10}^2 r_{20}\sin(3\theta_{10} - \theta_{20}) \\
a_{21} &= -\mu m_2[4r_{10}r_{20} + r_{10}^2\cos(3\theta_{10} - \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= -3a_{24} = \mu m_2 r_{10}^3\sin(3\theta_{10} - \theta_{20}) \\
a_{31} &= \mu m_1 r_{20}\sin(3\theta_{10} - \theta_{20}) \\
a_{32} &= \mu m_1 r_{10}\sin(3\theta_{10} - \theta_{20}) \\
a_{33} &= -3a_{34} = 3\mu m_1 r_{10}r_{20}\cos(3\theta_{10} - \theta_{20}) \\
a_{41} &= -\mu m_2\frac{r_{10}^2}{r_{20}}\sin(3\theta_{10} - \theta_{20}) \\
a_{42} &= \frac{1}{3}\mu m_2\frac{r_{10}^3}{r_{20}^2}\sin(3\theta_{10} - \theta_{20}) \\
a_{43} &= -3a_{44} = -\mu m_2\frac{r_{10}^3}{r_{20}}\cos(3\theta_{10} - \theta_{20})
\end{aligned}
\tag{5.33}
$$

Since $a_{i4} = -a_{i3}/3$ $(i = 1,.., 4)$ in Eqs. (5.33) the characteristic equation is reduced to the same form as Eq. (4.47) and the stability conditions are given by (4.48).

<u>Numerical Example</u>

We consider the case in which

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

and we assume that $\omega/n_1 = 0.4$. The amplitudes r_{10} and r_{20} of the combination oscillations are calculated by using Eqs. (5.28) and (5.29), and are plotted in Fig. 5.12. The fine and the thick lines show the combination oscillations corresponding to (2) and (3) of Eqs. (5.27), respectively. Two kinds of almost periodic oscillations are stably sustained.

(d) Entrained Oscillations Which Occur When $\omega \cong \omega_1/3 \cong \omega_2/9$

The steady-state solutions of Eqs. (5.3) are obtained by equating $\dot{r}_i = \dot{\theta}_i = 0$ $(i = 1, 2)$. Eliminating θ_{10} as before leads to

$$
\begin{aligned}
&[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2]r_{20}^2 = \frac{1}{9} r_{10}^6 \\
&[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\
&\quad + (\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)^2 = \frac{1}{9} A^6 r_{10}^2
\end{aligned}
\tag{5.34}
$$

where

$$
\left.
\begin{aligned}
\sigma_1 &= \frac{8n_1(k_2 - k_1)\omega_{11}}{k_2\omega_1^2} = \frac{8n_1(k_2 - k_1)}{\mu k_2 \omega_1^2}(\omega_1 - 3\omega) \\
\sigma_2 &= \frac{8n_1(k_1 - k_2)\omega_{21}}{k_1\omega_2^2} = \frac{8n_1(k_1 - k_2)}{\mu k_1 \omega_2^2}(\omega_2 - 9\omega)
\end{aligned}
\right\} \quad \text{detunings}
$$

Solving Eqs. (5.34) gives the amplitudes r_{10} and r_{20}. Then, the phase angles θ_{10} and θ_{20} are given by

$$
\begin{aligned}
\sin\theta_{10} &= -\frac{3}{A^3 r_{10}}(\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2) \\
\cos\theta_{10} &= \frac{3}{A^3 r_{10}}[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2] \\
\sin(3\theta_{10} - \theta_{20}) &= \frac{3\sigma_2 r_{20}}{r_{10}^3}
\end{aligned}
\tag{5.35}
$$

$$\cos(3\theta_{10} - \theta_{20}) = \frac{3r_{20}}{r_{10}^3}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)$$

The periodic solution corresponds to the oscillation with three frequencies ω, $3\omega (= \omega_{10})$, and $9\omega (= \omega_{20})$.

The stability of the steady-state solutions is tested as before.

(e) Entrained Oscillations Which Occur When $\omega \cong 2\omega_1 \cong 2\omega_2/3$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ in Eqs. (5.4), we obtain two types of the steady states, i.e.,

$$\begin{aligned} &(1) \quad r_{10} = 0, \quad r_{20} = 0 \\ &(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0 \end{aligned} \tag{5.36}$$

In the first case, the solution (4.16) is periodic of the frequency ω. Hence the harmonic entrainment occurs. In the second case, the solution (4.16) is periodic of the frequencies ω, $\omega/2$ $(= \omega_{10})$, and $3\omega/2$ $(= \omega_{20})$.

In the same manner as before, the amplitudes r_{10} and r_{20} are obtained by solving

$$\begin{aligned} &[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\ &\quad + \frac{1}{4}(\sigma_1 r_{10}^2 - \sigma_2 r_{20}^2)^2 = \frac{4}{9} r_{10}^6 r_{20}^2 \\ &[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\ &\quad + \frac{1}{4}(\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)^2 = 4A_1^4 r_{10}^2 r_{20}^2 \end{aligned} \tag{5.37}$$

where

$$\left.\begin{aligned} \sigma_1 &= \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - \omega/2) \\ \sigma_2 &= \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - 3\omega/2) \end{aligned}\right\} \text{detunings}$$

The phase angles θ_{10} and θ_{20} are given by

$$\sin(\theta_{10} + \theta_{20}) = -\frac{1}{4A_1^2 r_{10} r_{20}} (\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)$$

$$\cos(\theta_{10} + \theta_{20}) = \frac{1}{2A_1^2 r_{10} r_{20}} [3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2) r_{20}^2 - (\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) r_{10}^2]$$

$$\sin(3\theta_{10} - \theta_{20}) = \frac{3}{4r_{10}^3 r_{20}} (-\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2)$$

$$\cos(3\theta_{10} - \theta_{20}) = \frac{3}{2r_{10}^3 r_{20}} [(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2) r_{20}^2] \tag{5.38}$$

Stability Investigation

The stability of the steady state (2) is tested as before. The coefficients of the variational equations are as follows:

$$a_{11} = \mu m_1 [\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2 - 2r_{10} r_{20} \cos(3\theta_{10} - \theta_{20})]$$

$$a_{12} = -\mu m_1 [4r_{10} r_{20} + r_{10}^2 \cos(3\theta_{10} - \theta_{20}) + A_1^2 \cos(\theta_{10} + \theta_{20})]$$

$$a_{13} = \mu m_1 [3r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) + A_1^2 r_{20} \sin(\theta_{10} + \theta_{20})]$$

$$a_{14} = \mu m_1 [-r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) + A_1^2 r_{20} \sin(\theta_{10} + \theta_{20})]$$

$$a_{21} = -\mu m_2 [4r_{10} r_{20} + r_{10}^2 \cos(3\theta_{10} - \theta_{20}) + A_1^2 \cos(\theta_{10} + \theta_{20})]$$

$$a_{22} = \mu m_2 (\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2)$$

$$a_{23} = \mu m_2 [r_{10}^3 \sin(3\theta_{10} - \theta_{20}) + A_1^2 r_{10} \sin(\theta_{10} + \theta_{20})]$$

$$a_{24} = \mu m_2 [-\frac{1}{3} r_{10}^3 \sin(3\theta_{10} - \theta_{20}) + A_1^2 r_{10} \sin(\theta_{10} + \theta_{20})]$$

$$a_{31} = \mu m_1[r_{20} \sin (3\theta_{10} - \theta_{20}) - \frac{A_1^2 r_{20}}{r_{10}^2} \sin (\theta_{10} + \theta_{20})]$$

$$a_{32} = \mu m_1[r_{10} \sin (3\theta_{10} - \theta_{20}) + \frac{A_1^2}{r_{10}} \sin (\theta_{10} + \theta_{20})] \tag{5.39}$$

$$a_{33} = \mu m_1[3r_{10}r_{20} \cos (3\theta_{10} - \theta_{20}) + \frac{A_1^2 r_{20}}{r_{10}} \cos (\theta_{10} + \theta_{20})]$$

$$a_{34} = \mu m_1[- r_{10}r_{20} \cos (3\theta_{10} - \theta_{20}) + \frac{A_1^2 r_{20}}{r_{10}} \cos (\theta_{10} + \theta_{20})]$$

$$a_{41} = \mu m_2[- \frac{r_{10}^2}{r_{20}} \sin (3\theta_{10} - \theta_{20}) + \frac{A_1^2}{r_{20}} \sin (\theta_{10} + \theta_{20})]$$

$$a_{42} = \mu m_2[\frac{1}{3} \frac{r_{10}^3}{r_{20}^2} \sin (3\theta_{10} - \theta_{20}) - \frac{A_1^2 r_{10}}{r_{20}^2} \sin (\theta_{10} + \theta_{20})]$$

$$a_{43} = \mu m_2[- \frac{r_{10}^3}{r_{20}} \cos (3\theta_{10} - \theta_{20}) + \frac{A_1^2 r_{10}}{r_{20}} \cos (\theta_{10} + \theta_{20})]$$

$$a_{44} = \mu m_2[\frac{1}{3} \frac{r_{10}^3}{r_{20}} \cos (3\theta_{10} - \theta_{20}) + \frac{A_1^2 r_{10}}{r_{20}} \cos (\theta_{10} + \theta_{20})]$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

In this section (e) we are dealing the case $\omega = 2\omega_{10} = \frac{2}{3}\omega_{20}$. Therefore we obtain $\omega = \frac{1}{2}(\omega_{10} + \omega_{20})$. Under this condition the stability of the steady state (1) $[r_{10} = 0,\ r_{20} = 0]$ is discussed in the same manner as in Sec. 4.3.5 and the stability conditions are given by (4.91).

Numerical Example

By use of Eqs. (5.37) the amplitude characteristics (r_{10}, r_{20} vs B) are calculated for

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

and $\omega = 1.491n_1$; $(\sigma_1 = \sigma_2 = 0)$. The result is plotted in Fig. 5.13. The stability of the steady states is discussed as before. The dotted portions of the characteristic curves represent the unstable states. We see, in the figures, that two kinds of entrained oscillations (a and c) exist. It depends on the initial condition as regards which kind of oscillations occurs.

(f) Entrained Oscillations Which Occur When $\hat{\omega} \cong 5\omega_1 \cong 5\omega_2/3$

The steady-state solutions of Eqs. (5.5) are obtained by solving

$$\begin{aligned}
&(\rho_1 - 2A_1 - r_{10}^2 - 2r_{20}^2)r_{10} - r_{10}^2 r_{20} \cos(3\theta_{10} - \theta_{20}) \\
&\quad - A_1 r_{20}^2 \cos(\theta_{10} - 2\theta_{20}) - 2A_1 r_{10} r_{20} \cos(2\theta_{10} + \theta_{20}) = 0 \\
&(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20} - \frac{1}{3} r_{10}^3 \cos(3\theta_{10} - \theta_{20}) \\
&\quad - 2A_1 r_{10} r_{20} \cos(\theta_{10} - 2\theta_{20}) - A_1 r_{10}^2 \cos(2\theta_{10} + \theta_{20}) = 0 \\
&\omega_{11} r_1 + \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)} [r_{10}^2 r_{20} \sin(3\theta_{10} - \theta_{20}) \\
&\quad + A_1 r_{20}^2 \sin(\theta_{10} - 2\theta_{20}) + 2A_1 r_{10} r_{20} \sin(2\theta_{10} + \theta_{20})] = 0 \\
&\omega_{21} r_{20} - \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)} [\frac{1}{3} r_{10}^3 \sin(3\theta_{10} - \theta_{20}) \\
&\quad + 2A_1 r_{10} r_{20} \sin(\theta_{10} - 2\theta_{20}) - A_1 r_{10}^2 \sin(2\theta_{10} + \theta_{20})] = 0
\end{aligned} \tag{5.40}$$

From Eqs. (5.40) we see that there are two types of steady states

$$\begin{aligned}
&(1) \quad r_{10} = 0, \quad r_{20} = 0 \\
&(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0
\end{aligned} \tag{5.41}$$

The first case is the harmonic entrainment which is already discussed in (c) and (d) of this section. The second case corresponds to the periodic oscillation with three frequencies ω, $\omega/5$ ($=\omega_{10}$), and $3\omega/5$ ($=\omega_{20}$). In this case, however, we cannot eliminate the phase angles θ_{10} and θ_{20} from Eqs. (5.40), because of the existence of three frequency components $3\theta_{10}-\theta_{20}$, $\theta_{10}-2\theta_{20}$, and $2\theta_{10}+\theta_{20}$. Solving Eqs. (5.40) simultaneously gives r_{10}, r_{20}, θ_{10}, and θ_{20}. This procedure of solution, however, is not simple in practice. It is convenient to introduce the rectangular co-ordinates a_i, b_i $(i = 1, 2)$ instead of the polar co-ordinates r_i, θ_i of Eqs. (4.6), i.e.,

$$\begin{aligned} x(t) &= a_1(t)\cos\omega_1 t + b_1(t)\sin\omega_1 t \\ y(t) &= a_2(t)\cos\omega_2 t + b_2(t)\sin\omega_2 t \end{aligned} \tag{5.42}$$

By comparing Eqs. (5.42) with Eqs. (4.6), we obtain

$$\begin{aligned} a_1(t) &= r_1(t)\cos\theta_1(t), \qquad & b_1(t) &= -r_1(t)\sin\theta_1(t) \\ a_2(t) &= r_2(t)\cos\theta_2(t), \qquad & b_2(t) &= -r_2(t)\sin\theta_2(t) \end{aligned} \tag{5.43}$$

Using Eqs. (5.5) and (5.43), we obtain the averaged equations with respect to a_i and b_i $(i = 1, 2)$, i.e.,

$$\begin{aligned} \dot{a}_1 &= \frac{\mu\omega_1^2 k_2}{8n_1(k_2-k_1)}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)a_1 - (a_1^2-b_1^2)a_2 - 2a_1b_1b_2 \\ &\quad - 2A_1(a_1a_2 - b_1b_2) - A_1(a_2^2 - b_2^2)] + \mu\omega_{11}b_1 \\ \dot{b}_1 &= \frac{\mu\omega_1^2 k_2}{8n_1(k_2-k_1)}[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)b_1 - (a_1^2-b_1^2)b_2 + 2a_1b_1a_2 \\ &\quad + 2A_1(a_1b_2 + b_1a_2) - 2A_1a_2b_2] - \mu\omega_{11}a_1 \\ \dot{a}_2 &= \frac{\mu\omega_2^2 k_1}{8n_1(k_1-k_2)}[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)a_2 - \frac{4}{3}a_1^3 + r_1^2a_1 \\ &\quad - 2A_1(a_1a_2 + b_1b_2) - A_1(a_1^2 - b_1^2)] + \mu\omega_{21}b_2 \end{aligned} \tag{5.44}$$

$$\dot{b}_2 = \frac{\mu\omega_2^2 k_1}{8n_1(k_1 - k_2)} [(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)b_2 + \frac{4}{3} b_1^3 - r_1^2 b_1$$

$$+ 2A_1(a_1b_2 - b_1a_2) + 2A_1a_1b_1] - \mu\omega_{21}a_2$$

where

$$r_1^2 = a_1^2 + b_1^2, \qquad r_2^2 = a_2^2 + b_2^2$$

Equating $\dot{a}_i = \dot{b}_i = 0$ $(i = 1, 2)$ in Eqs. (5.44) and solving them gives the steady-state solutions where $r_{10} \neq 0$ and $r_{20} \neq 0$.

Stability Investigation

Since the first case of Eqs. (5.41) is identical with that of Eqs. (4.54), the stability conditions are also given by (4.56). The stability of the second case is tested by deriving the variational equations from Eqs. (5.44) and by making use of the Routh-Hurwitz criterion. The conditions are given by the same form as (4.46), where the coefficients of the variational equations are as follows:

$$a_{11} = \mu m_1(\rho_1 - 2A_1^2 - 3a_{10}^2 - b_{10}^2 - 2r_{20}^2 - 2a_{10}a_{20} - 2b_{10}b_{20} - 2A_1a_{20})$$

$$a_{12} = 2\mu m_1(- a_{10}b_{10} + b_{10}a_{20} - a_{10}b_{20} + A_1b_{20}) + \mu\omega_{11}$$

$$a_{13} = \mu m_1(- 4a_{10}a_{20} - a_{10}^2 + b_{10}^2 - 2A_1a_{10} - 2A_1a_{20})$$

$$a_{14} = 2\mu m_1(- a_{10}b_{10} - 2a_{10}b_{20} + A_1b_{10} + A_1b_{20})$$

$$a_{21} = 2\mu m_1(- a_{10}b_{10} - a_{10}b_{20} + b_{10}a_{20} + A_1b_{20}) - \mu\omega_{11}$$

$$a_{22} = \mu m_1(\rho_1 - 2A_1^2 - a_{10}^2 - 3b_{10}^2 - 2r_{20}^2 + 2a_{10}a_{20} + 2b_{10}b_{20} + 2A_1a_{20})$$

$$a_{23} = 2\mu m_1(a_{10}b_{10} - 2b_{10}a_{20} + A_1b_{10} - A_1b_{20})$$

$$a_{24} = \mu m_1(- a_{10}^2 - 4b_{10}b_{20} + b_{10}^2 + 2A_1a_{10} - 2A_1a_{20})$$

$$a_{31} = \mu m_2(- a_{10}^2 + b_{10}^2 - 4a_{10}a_{20} - 2A_1a_{10} - 2A_1a_{20})$$

$$a_{32} = \mu m_2(-\ 2a_{10}b_{10} - 4b_{10}a_{20} + 2A_1b_{10} - 2A_1b_{20}) \tag{5.45}$$

$$a_{33} = \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - 3a_{20}^2 - b_{20}^2 - 2A_1a_{10})$$

$$a_{34} = -\ 2\mu m_2(a_{20}b_{20} + A_1b_{10}) + \mu\omega_{21}$$

$$a_{41} = 2\mu m_2(-\ a_{10}b_{10} - 2a_{10}b_{20} + A_1b_{10} + A_1b_{20})$$

$$a_{42} = \mu m_2(-\ a_{10}^2 + b_{10}^2 - 4b_{10}b_{20} + 2A_1a_{10} - 2A_1a_{20})$$

$$a_{43} = -\ 2\mu m_2(a_{20}b_{20} + A_1b_{10}) - \mu\omega_{21}$$

$$a_{44} = \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - a_{20}^2 - 3b_{20}^2 + 2A_1a_{10})$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

Numerical Example

A numerical analysis of the response characteristic for the entrained oscillation is carried out by using the same parameters of the system as before, i.e.,

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

and we assume that $\omega = 5\omega_1 = 5\omega_2/3 = 3.727n_1\,(\sigma_1 = \sigma_2 = 0)$. The response characteristics calculated by using Eqs. (5.44) are illustrated in Fig. 5.14. We see that there are two stable steady states and that the hysteresis occurs as B varies.

The theoretical results thus obtained are compared with the solutions obtained by using an analog computer. Some representative waveforms u(t) and v(t) are shown in Fig. 5.15a. By using Eqs. (1.20) the waveforms of x(t) and y(t) are obtained from u(t) and v(t). These waveforms are shown in Fig. 5.15b.

We see that the waveforms of x and y are nearly sinusoidal. This fact shows that the assumption of the solution (4.6) is permissive. The amplitude characteristics (r_{10}, r_{20}) obtained from the waveforms of x and y are shown in Fig. 5.16. These characteristics agree well with the theoretical results of Fig. 5.14. The result concerning the hysteresis mentioned above is also confirmed by an analog computer analysis (see Figs. 5.14 and 5.16).

(g) Entrained Oscillations Which Occur When $\omega \cong 7\omega_1 \cong 7\omega_2/3$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ in Eqs. (5.6), we obtain two types of the steady-state solutions, i.e.,

$$\begin{aligned} &(1)\quad r_{10} = 0,\quad r_{20} = 0 \\ &(2)\quad r_{10} \neq 0,\quad r_{20} \neq 0 \end{aligned} \tag{5.46}$$

The first case is a harmonic oscillation and its stability conditions are also identical with (4.56). In the second case, the solution is a periodic oscillation with frequencies ω, $\omega/7$ ($= \omega_{10}$), and $3\omega/7$ ($= \omega_{20}$). The amplitudes r_{10} and r_{20} are obtained by solving

$$\begin{aligned} &\frac{9}{25}\left[2(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2\right]^2 \\ &\quad + \frac{9}{49}\left[2\sigma_1 r_{10}^2 - \sigma_2 r_{20}^2\right]^2 = r_{10}^6 r_{20}^2 \\ &\frac{1}{25}\left[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2\right]^2 \\ &\quad + \frac{1}{49}(\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)^2 = A_1^2 r_{10}^2 r_{20}^4 \end{aligned} \tag{5.47}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}\left(\omega_1 - \frac{\omega}{7}\right)$$

detunings

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - \frac{3}{7}\omega)$$

The phase angles θ_{10} and θ_{20} are given by

$$\begin{aligned}
\sin(3\theta_{10} - \theta_{20}) &= -\frac{3}{7r_{10}^3 r_{20}}(2\sigma_1 r_{10}^2 - \sigma_2 r_{20}^2) \\
\cos(3\theta_{10} - \theta_{20}) &= \frac{3}{5r_{10}^3 r_{20}}[2(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 \\
&\quad - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2] \\
\sin(\theta_{10} + 2\theta_{20}) &= -\frac{1}{7A_1 r_{10} r_{20}^2}(\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2) \\
\cos(\theta_{10} + 2\theta_{20}) &= -\frac{1}{5A_1 r_{10} r_{20}^2}[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 \\
&\quad - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]
\end{aligned} \tag{5.48}$$

The stability of the steady-state solutions is tested as before.

Numerical Example

We consider the same values of parameters as in the preceding sections, i.e.,

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

and we assume that $\omega = 7\omega_1 = 7\omega_2/3 = 5.218n_1$ (i.e. $\sigma_1 = \sigma_2 = 0$). The amplitude characteristics (r_{10}, r_{20} vs B) calculated by using Eqs. (5.47) are illustrated in Fig. 5.17.

(h) Entrained Oscillations Which Occur When $\omega \cong 9\omega_1 \cong 3\omega_2$

Equating $\dot{r}_i = \dot{\theta}_i = 0$ ($i = 1, 2$) in Eqs. (5.7) yields three types of steady states for Eqs. (4.16), i.e.,

(1) $r_{10} = 0, \quad r_{20} = 0$

$$(2) \quad r_{10} = 0, \quad r_{20} \neq 0 \tag{5.49}$$

$$(3) \quad r_{10} \neq 0, \quad r_{20} \neq 0$$

The first and the second cases of Eqs. (5.49) show the harmonic and subharmonic entrainments, respectively. In the second case the amplitude r_{20} and the phase angle θ_{20} are given by

$$(\rho_2 - 2A_1^2 - r_{20}^2)^2 + \sigma_2^2 = A_1^2 r_{20}^2$$

where

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - \omega/3) \ : \text{detuning}$$

$$\sin 3\theta_{20} = -\frac{\sigma_2}{A_1 r_{20}} \tag{5.50}$$

$$\cos 3\theta_{20} = \frac{1}{A_1 r_{20}}(\rho_2 - 2A_1^2 - r_{20}^2)$$

By comparing Eqs. (5.50) with Eqs. (4.71), we see that they are equivalent. Therefore, the stability conditions for the first and the second cases of Eqs. (5.49) are analogous to those of (4.75) and (4.76).

The steady state (3) of Eqs. (5.49) is a periodic solution with three frequencies ω, $\omega/3$ ($= \omega_{10}$), and $\omega/9$ ($= \omega_{20}$). The amplitudes r_{10} and r_{20} are obtained by solving

$$(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_1^2 = r_{10}^2 r_{20}^2$$

$$[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 3(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2$$

$$+ (\sigma_1 r_{10}^2 + 3\sigma_2 r_{20}^2)^2 = 9A_1^2 r_{20}^6 \tag{5.51}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\omega_1^2 k_2}(\omega_1 - \omega/9) \ : \text{detuning}$$

The phase angles θ_{10} and θ_{20} are given by

$$
\begin{aligned}
\sin\,(3\theta_{10} - \theta_{20}) &= -\,\sigma_1 r_{10} r_{20} \\
\cos\,(3\theta_{10} - \theta_{20}) &= \frac{1}{r_{10} r_{20}}\,(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) \\
\sin\,3\theta_{20} &= -\,\frac{1}{A_1 r_{20}^3}\,(\tfrac{1}{3}\,\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2) \\
\cos\,3\theta_{20} &= \frac{1}{A_1 r_{20}^3}\,[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2) r_{20}^2 \\
&\qquad - \tfrac{1}{3}\,(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2) r_{10}^2]
\end{aligned}
\tag{5.52}
$$

The stability is tested as before.

<u>Numerical Example</u>

Let us consider the case where

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

and we assume that $\omega = 9\omega_1 = 3\omega_2 = 6.708 n_1$ (i.e. $\sigma_1 = \sigma_2 = 0$). The amplitude characteristics (r_{10}, r_{20} vs B) calculated by using Eqs. (5.51) are illustrated in Fig. 5.18. For comparison's sake, the amplitude r_{20} of the 1/3 - harmonic oscillation is calculated by using Eqs. (5.50) and shown in the figure by the fine lines.

(i) Regions of Frequency Entrainment

From the results obtained in this section 5.3, the regions of frequency entrainment are produced on the $B\omega$ plane. Figure 5.19 shows the case in which the system parameters are the same as before, i.e.,

$$\mu = 0.2 \qquad k = 0.8 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0$$

One sees that the harmonic entrainment occurs at any driving frequency ω

provided that the amplitude B of the external force is sufficiently large. The continuity of the boundary curve of the harmonic entrainment is disturbed by the intrusion of the region of the frequency entrainment which occurs when $\omega = 2\omega_1 = 2\omega_2/3$.

The frequency entrainment also occurs within a narrow range of the driving frequency ω, when ω is in the neighborhood of 1/3, 2, 5, 7, or 9 times the natural frequency ω_1 (= $\omega_2/3$). As mentioned before, more than two types of the

Table 5.1 The frequency components containd in the steady-state solutions (internal resonance : $3\omega_{10} = \omega_{20}$)

Type	ω is in the neighborhood of ($k = 0.8$, $n_2/n_1 = 1.0$)	Steady states			
		(1) $r_{10} = 0$, $r_{20} = 0$	(2) $r_{10} \neq 0$, $r_{20} = 0$	(3) $r_{10} = 0$, $r_{20} \neq 0$	(4) $r_{10} \neq 0$, $r_{20} \neq 0$
B3	$\frac{1}{3}\omega_1 = \frac{1}{9}\omega_2 = 0.248n_1$	——	——	——	ω, 3ω, 9ω
B1	$\omega_1 = \frac{1}{3}\omega_2 = 0.745n_1$	——	——	——	ω, 3ω
B4	$2\omega_1 = \frac{2}{3}\omega_2 = 1.491n_1$	ω	——	——	ω, $\frac{1}{2}\omega$ $\frac{3}{2}\omega$
B2	$3\omega_1 = \omega_2 = 2.236n_1$	——	——	ω	ω, $\frac{1}{3}\omega$
B5	$5\omega_1 = \frac{5}{3}\omega_2 = 3.727n_1$	ω	——	——	ω, $\frac{1}{5}\omega$, $\frac{3}{5}\omega$
B6	$7\omega_1 = \frac{7}{3}\omega_2 = 5.218n_1$	ω	——	——	ω, $\frac{1}{7}\omega$, $\frac{3}{7}\omega$
B7	$9\omega_1 = 3\omega_2 = 6.708n_1$	ω	——	ω, $\frac{1}{3}\omega$	ω, $\frac{1}{3}\omega$, $\frac{1}{9}\omega$
B8	Non-Resonant Case	ω	——	ω, ω_2	ω, ω_{10}^{*}, $3\omega_{10}(=\omega_{20})$

* Note : ω_{10} is the frequency of the self-excited oscillation.

entrained oscillations may occur in these regions. The details of the regions for each entrained oscillation are not shown in the figure. By making use of the results in this section, the types of the internal and external resonances, and the frequency components of the entrained oscillations in the steady state are summarized in Table 5.1. The steady state (4) in the Table corresponds to the entrained oscillation by the existence of the internal resonance.

5.4 Entrained Oscillations in a System with Internal Resonance $m\omega_1 \cong n\omega_2$ (m, n : positive integers)

In this section we consider forced oscillations in a system whose two natural frequencies have a relationship $m\omega_1 = n\omega_2$, where m and n are positive integers ($s \equiv n/m \neq 1$ or 3).* When the driving frequency ω is chosen as listed in Table 4.3, more than two types of the external resonance occurs in this system (see Sec. 4.3). Each natural frequency is entrained by the frequency which is an integral multiple or submultiple of the driving frequency. Hence the resulting oscillation is periodic with three frequency components. We discuss the steady-state solutions and their stability by the same procedure as in the preceding section.

(a) Entrained Oscillations Which Occur When $\omega \cong \omega_1/2 \cong \omega_2/3$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ ($i = 1, 2$) in Eqs. (5.9), we see that there are two types of the steady states, i.e.,

$$(1) \quad r_{10} = 0, \quad r_{20} \neq 0$$

---

* The internal resonance does not occur in this system if the external force is absent (see Chap. 3).

$$(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0 \tag{5.53}$$

The first case corresponds to the subharmonic entrainment which was discussed in Sec. 4.3.4. In the second case, the solution (4.16) is periodic of the frequencies ω, $2\omega (= \omega_{10})$, and $3\omega (= \omega_{20})$. In the same manner as in the preceding section, the amplitudes r_{10} and r_{20} are obtained by solving

$$\begin{aligned}
&(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_1^2 = 4A_1^2 r_{20}^2 \\
&[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\
&\quad + (\sigma_1 r_{10}^2 + 2\sigma_2 r_{20}^2)^2 = \frac{4}{9} A_1^6 r_{20}^2
\end{aligned} \tag{5.54}$$

where

$$\left.\begin{aligned}
\sigma_1 &= \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - \omega/2) \\
\sigma_2 &= \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_1 - 3\omega/2)
\end{aligned}\right\} \text{detunings}$$

The phase angles θ_{10} and θ_{20} are given by

$$\begin{aligned}
\sin\theta_{20} &= -\frac{3}{2A_1^3 r_{20}}(\sigma_1 r_{10}^2 + 2\sigma_2 r_{20}^2) \\
\cos\theta_{20} &= \frac{3}{2A_1^3 r_{20}}[2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2 - (\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2] \\
\sin(2\theta_{10} - \theta_{20}) &- \frac{\sigma_1}{2A_1 r_{20}} \\
\cos(2\theta_{10} - \theta_{20}) &= \frac{1}{2A_1 r_{20}}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)
\end{aligned} \tag{5.55}$$

Stability Investigation

The stability of the steady-state solutions is tested in the same manner as before. The variational equations are sought from Eqs. (5.9) [see Eqs.

4.42)]. Their coefficients a_{ij} are given by

$$
\begin{aligned}
a_{11} &= \mu m_1[\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2 - 2A_1 r_{20}\cos(2\theta_{10} - \theta_{20})] \\
a_{12} &= -2\mu m_1[2r_{10}r_{20} + A_1 r_{10}\cos(2\theta_{10} - \theta_{20})] \\
a_{13} &= 4\mu m_1 A_1 r_{10} r_{20}\sin(2\theta_{10} - \theta_{20}) \\
a_{14} &= -2\mu m_1 A_1 r_{10} r_{20}\sin(2\theta_{10} - \theta_{20}) \\
a_{21} &= -2\mu m_2[2r_{10}r_{20} + A_1 r_{10}\cos(2\theta_{10} - \theta_{20})] \\
a_{22} &= \mu m_2(\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2) \\
a_{23} &= 2\mu m_2 A_1 r_{10}^2\sin(2\theta_{10} - \theta_{20}) \\
a_{24} &= \mu m_2[\tfrac{1}{3}A_1^3\sin\theta_{20} - A_1 r_{10}^2\sin(2\theta_{10} - \theta_{20})] \\
a_{31} &= 0 \\
a_{32} &= 2\mu m_1 A_1\sin(2\theta_{10} - \theta_{20}) \\
a_{33} &= 4\mu m_1 A_1 r_{20}\cos(2\theta_{10} - \theta_{20}) \\
a_{34} &= -2\mu m_1 A_1 r_{20}\cos(2\theta_{10} - \theta_{20}) \\
a_{41} &= -2\mu m_2\frac{A_1 r_{10}}{r_{20}}\sin(2\theta_{10} - \theta_{20}) \\
a_{42} &= \mu m_2[-\frac{1}{3}\frac{A_1^3}{r_{20}^2}\sin\theta_{20} + \frac{A_1 r_{10}^2}{r_{20}}\sin(2\theta_{10} - \theta_{20})] \\
a_{43} &= -2\mu m_2\frac{A_1 r_{10}^2}{r_{20}}\cos(2\theta_{10} - \theta_{20}) \\
a_{44} &= \mu m_2[\frac{1}{3}\frac{A_1^3}{r_{20}}\cos\theta_{20} + \frac{A_1 r_{10}^2}{r_{20}}\cos(2\theta_{10} - \theta_{20})]
\end{aligned}
\tag{5.56}
$$

where

$$
m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}
$$

Using these values of the coefficients, we obtain the stability conditions (4.46).

(b) Entrained Oscillations Which Occur When $\omega \cong \omega_1/3 \cong \omega_2/5$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ ($i = 1, 2$) in Eqs. (5.10), we obtain only one steady state where $r_{10} \neq 0$, $r_{20} \neq 0$. In this case the solution (4.16) is periodic of the frequencies ω, 3ω ($= \omega_{10}$), and 5ω ($= \omega_{20}$). We cannot eliminate θ_{10} and θ_{20} from the equations $\dot{r}_i = 0$ and $\dot{\theta}_i = 0$. Introducing new variables a_i and b_i ($i = 1, 2$) by Eqs. (5.43) as in Sec. 5.3(f), we obtain the following autonomous equations:

$$\dot{a}_1 = \frac{\mu\omega_1^2 k_2}{8n_1(k_2 - k_1)}\left[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)a_1 - \frac{1}{3}A_1^3 - A_1^2 a_2 - 2A_1(a_1a_2 + b_1b_2) + \sigma_1 b_1\right]$$

$$\dot{b}_1 = \frac{\mu\omega_1^2 k_2}{8n_1(k_2 - k_1)}\left[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)b_1 - A_1^2 b_2 + 2A_1(a_1b_2 - b_1a_2) - \sigma_1 a_1\right]$$

$$\dot{a}_2 = \frac{\mu\omega_2^2 k_1}{8n_1(k_1 - k_2)}\left[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)a_2 - A_1^2 a_1 - A_1(a_1^2 - b_1^2) + \sigma_2 b_2\right] \tag{5.57}$$

$$\dot{b}_2 = \frac{\mu\omega_2^2 k_1}{8n_1(k_1 - k_2)}\left[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)b_2 - A_1^2 b_1 - 2A_1 a_1 b_1 - \sigma_2 a_2\right]$$

where

$$r_1^2 = a_1^2 + b_1^2, \qquad r_2^2 = a_2^2 + b_2^2$$

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - 3\omega) \qquad \text{detunings}$$

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - 5\omega)$$

By equating $\dot{a}_i = \dot{b}_i = 0$ ($i = 1, 2$) in Eqs. (5.57) and solving them we obtain the amplitudes a_{i0} and b_{i0}. The stability of this steady-state solution is tested as before.

(c) Entrained Oscillations Which Occur When $\omega \cong 3\omega_1 \cong 3\omega_2/2$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ in Eqs. (5.11), we find that there are three types of steady states, i.e.,

$$\begin{aligned} &(1) \quad r_{10} = 0, \quad r_{20} = 0 \\ &(2) \quad r_{10} \neq 0, \quad r_{20} = 0 \\ &(3) \quad r_{10} \neq 0, \quad r_{20} \neq 0 \end{aligned} \tag{5.58}$$

In case (1) the harmonic entrainment occurs, while in case (2) the subharmonic entrainment results. They were discussed in (1) and (2) of Eqs. (4.70). In case (3), the solution (4.16) is periodic of three frequencies ω, $\omega/3$ ($= \omega_{10}$), and $2\omega/3$ ($= \omega_{20}$). The amplitudes r_{10} and r_{20} of the solution (4.16) are obtained by solving

$$\begin{aligned} &(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2 = 4A_1^2 r_{10}^2 \\ &[2(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\ &\quad + (2\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2)^2 = 4A_1^2 r_{10}^6 \end{aligned} \tag{5.59}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\omega_1^2 k_2}(\omega_1 - \frac{\omega}{3})$$

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - \frac{2}{3}\omega) \qquad \text{detunings}$$

The phase angles θ_{10} and θ_{20} are given by

$$\sin 3\theta_{10} = -\frac{1}{2A_1 r_{10}^3}(2\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2)$$

$$\cos 3\theta_{10} = \frac{1}{2A_1 r_{10}^3}[2(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]$$

$$\sin(\theta_{10} - 2\theta_{20}) = \frac{\sigma_2}{2A_1 r_{10}} \tag{5.60}$$

$$\cos(\theta_{10} - 2\theta_{20}) = \frac{1}{2A_1 r_{10}}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)$$

Stability Investigation

The stability of the steady-state solution is tested as before. The coefficients of the variational equations are

$$a_{11} = \mu m_1(\rho_1 - 2A_1^2 - 3r_{10}^2 - 2r_{20}^2 - 2A_1 r_{10}\cos 3\theta_{10})$$

$$a_{12} = -2\mu m_1[2r_{10}r_{20} + A_1 r_{20}\cos(\theta_{10} - 2\theta_{20})]$$

$$a_{13} = \mu m_1[3A_1 r_{10}^2 \sin 3\theta_{10} + A_1 r_{20}^2 \sin(\theta_{10} - 2\theta_{20})]$$

$$a_{14} = -2\mu m_1 A_1 r_{20}^2 \sin(\theta_{10} - 2\theta_{20})$$

$$a_{21} = -2\mu m_2[2r_{10}r_{20} + A_1 r_{20}\cos(\theta_{10} - 2\theta_{20})]$$

$$a_{22} = \mu m_2[\rho_2 - 2A_1^2 - 2r_{10}^2 - 3r_{20}^2 - 2A_1 r_{10}\cos(\theta_{10} - 2\theta_{20})]$$

$$a_{23} = 2\mu m_2 A_1 r_{10} r_{20}\sin(\theta_{10} - 2\theta_{20})$$

$$a_{24} = -4\mu m_2 A_1 r_{10} r_{20}\sin(\theta_{10} - 2\theta_{20})$$

$$a_{31} = \mu m_1[A_1 \sin 3\theta_{10} - \frac{A_1 r_{20}^2}{r_{10}^2}\sin(\theta_{10} - 2\theta_{20})] \tag{5.61}$$

$$a_{32} = 2\mu m_1 \frac{A_1 r_{20}}{r_{10}} \sin(\theta_{10} - 2\theta_{20})$$

$$a_{33} = \mu m_1 [3A_1 r_{10} \cos 3\theta_{10} + \frac{A_1 r_{20}^2}{r_{10}} \cos(\theta_{10} - 2\theta_{20})]$$

$$a_{34} = -2\mu m_1 \frac{A_1 r_{20}^2}{r_{10}} \cos(\theta_{10} - 2\theta_{20})$$

$$a_{41} = -2\mu m_2 A_1 \sin(\theta_{10} - 2\theta_{20})$$

$$a_{42} = 0$$

$$a_{43} = -2\mu m_2 A_1 r_{10} \cos(\theta_{10} - 2\theta_{20})$$

$$a_{44} = 4\mu m_2 A_1 r_{10} \cos(\theta_{10} - 2\theta_{20})$$

where

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

Numerical Example

Let us consider the case in which

$$\mu = 0.2 \qquad k = 0.6 \qquad (n_2/n_1)^2\delta = 0.5$$

We find that $2\omega_1 = \omega_2$ provided $n_2/n_1 = 1.0$. We assume that $\omega = 3\omega_1 = 3\omega_2/2 = 2.372n_1$ (i.e. $\sigma_1 = \sigma_2 = 0$). The response characteristics calculated by using Eqs. (5.59) are shown in Fig. 5.9 (thick lines). The characteristic of 1/3-harmonic entrainment is also shown in the figure (fine lines). We see that two kinds of entrained oscillations are stable (a and f). It depends on the initial condition as regards which kind of the oscillations occurs.

(d) Entrained Oscillations Which Occur When $\omega \cong \omega_1/3 \cong \omega_2/7$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ $(i = 1, 2)$ in Eqs. (5.12), we obtain only one steady state where $r_{10} \neq 0$, $r_{20} \neq 0$. Therefore, the solution (4.16) is a periodic oscillation with three frequencies ω, 3ω (= ω_{10}), and 7ω (= ω_{20}). The amplitudes r_{10} and r_{20} of the solution (4.16) are obtained by solving

$$[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2]r_{20}^2 = A_1^2 r_{10}^4$$

$$[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2$$

$$+ (\sigma_1 r_{10}^2 + 2\sigma_2 r_{20}^2)^2 = \frac{1}{9} A_1^6 r_{10}^2 \tag{5.62}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - 3\omega)$$

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_1 - 7\omega)$$

The phase angles θ_{10} and θ_{20} are given by

$$\sin\theta_{10} = -\frac{3}{A_1^3 r_{10}}(\sigma_1 r_{10}^2 + 2\sigma_2 r_{20}^2)$$

$$\cos\theta_{10} = \frac{3}{A_1^3 r_{10}}[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - 2(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2] \tag{5.63}$$

$$\sin(2\theta_{10} - \theta_{20}) = \frac{\sigma_2 r_{20}}{A_1 r_{10}^2}$$

$$\cos(2\theta_{10} - \theta_{20}) = \frac{r_{20}}{A_1 r_{10}^2}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)$$

The stability of the steady-state solutions is tested as before.

(e) Entrained Oscillations Which Occur When $\omega \cong 3\omega_1 \cong 3\omega_2/5$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ ($i = 1, 2$) in Eqs. (5.13), we obtain two types of the steady states, i.e.,

(1) $r_{10} = 0$, $r_{20} = 0$

$$(2)\quad r_{10} \neq 0,\quad r_{20} \neq 0 \tag{5.64}$$

The first case is the harmonic entrainment. Since $\omega = 3\omega_{10} = 3\omega_{20}/5$, we obtain $\omega = (\omega_{10} + \omega_{20})/2$. Therefore, the stability conditions of this entrained oscillation is identical with (4.91). In the second case the solution (4.16) is periodic of the frequencies ω, $\omega/3$ ($= \omega_{10}$), and $5\omega/3$ ($= \omega_{20}$). In order to obtain the amplitudes r_{10} and r_{20} of the solution (4.16), it is convenient to introduce the rectangular co-ordinates by the same procedure as used in Sec. 5.3(f). We obtain the following autonomous equations:

$$\begin{aligned}
\dot{a}_1 &= \mu m_1[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)a_1 - A_1(a_1^2 - b_1^2) - A_1^2 a_2 \\
&\qquad - 2A_1(a_1 a_2 - b_1 b_2) + \sigma_1 b_1] \\
\dot{b}_1 &= \mu m_1[(\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2)b_1 + 2A_1 a_1 b_1 + A_1^2 b_2 \\
&\qquad - 2A_1(a_1 b_2 - b_1 a_2) - \sigma_1 a_1] \\
\dot{a}_2 &= \mu m_2[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)a_2 - A_1^2 a_1 - A_1(a_1^2 - b_1^2) + \sigma_2 b_2] \\
\dot{b}_2 &= \mu m_2[(\rho_2 - 2A_1^2 - 2r_1^2 - r_2^2)b_2 + A_1^2 b_1 - 2A_1 a_1 b_1 - \sigma_2 a_2]
\end{aligned} \tag{5.65}$$

where

$$r_1^2 = a_1^2 + b_1^2, \qquad r_2^2 = a_2^2 + b_2^2$$

$$m_1 = \frac{\omega_1^2 k_2}{8n_1(k_2 - k_1)}, \qquad m_2 = \frac{\omega_2^2 k_1}{8n_1(k_1 - k_2)}$$

$$\sigma_1 = \frac{\omega_{11}}{m_1} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2}(\omega_1 - \omega/3)$$

$$\sigma_2 = \frac{\omega_{21}}{m_2} = \frac{8n_1(k_1 - k_2)}{\omega_2^2 k_1}(\omega_2 - 5\omega/3)$$

By equating $\dot{a}_i = \dot{b}_i = 0$ ($i = 1, 2$) in Eqs. (5.65) and solving them, we obtain the amplitudes a_{10} and b_{10}.

The stability of this steady-state solution is tested as before.

(f) Entrained Oscillations Which Occur When $\omega \cong 3\omega_1 \cong 3\omega_2/7$

By equating $\dot{r}_i = \dot{\theta}_i = 0$ in Eqs. (5.14), we have two types of steady states.

$$\begin{aligned} &(1)\quad r_{10} = 0, \quad r_{20} = 0 \\ &(2)\quad r_{10} \neq 0, \quad r_{20} \neq 0 \end{aligned} \tag{5.66}$$

The first case is the harmonic entrainment. Since $\omega = 3\omega_{10} = 3\omega_{20}/7$, we obtain $\omega = (\omega_{20} - \omega_{10})/2$. The stability conditions of this entrained oscillation are given by (4.91) where ω_{11} must be replaced by $-\omega_{11}$. In the second case the solution (4.16) is periodic of the frequencies ω, $\omega/3$ ($= \omega_{10}$), and $7\omega/3$ ($= \omega_{20}$). The amplitudes r_{10} and r_{20} are obtained by solving the following equations:

$$\begin{aligned} &[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2]r_{20}^2 = A_1^4 r_{10}^2 \\ &[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]^2 \\ &\quad + (\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2)^2 = A_1^2 r_{10}^6 \end{aligned} \tag{5.67}$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_2^2 k_2}(\omega_1 - \omega/3)$$

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\omega_{21}}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_2^2 k_1}(\omega_2 - 7\omega/3)$$

The phase angles θ_{10} and θ_{20} are given by

$$\sin 3\theta_{10} = -\frac{1}{A_1 r_{10}^3}(\sigma_1 r_{10}^2 + \sigma_2 r_{20}^2)$$

$$\cos 3\theta_{10} = \frac{1}{A_1 r_{10}^3}[(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)r_{10}^2 - (\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)r_{20}^2]$$

$$\sin(\theta_{10} - \theta_{20}) = \frac{\sigma_2 r_{20}}{A_1^2 r_{10}} \tag{5.68}$$

$$\cos(\theta_{10} - \theta_{20}) = \frac{r_{20}}{A_1^2 r_{10}} (\rho_1 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)$$

The stability of this steady-state solution is tested as before.

(g) Entrained Oscillations Which Occur When $\omega \cong 3\omega_1 \cong \omega_2/3$

By equating $\dot{r}_1 = \dot{\theta}_1 = 0$ in Eqs. (5.13) we see that there are two types of the steady states.

$$\begin{aligned} &(1) \quad r_{10} = 0, \quad r_{20} \neq 0 \\ &(2) \quad r_{10} \neq 0, \quad r_{20} \neq 0 \end{aligned} \tag{5.69}$$

In the first case, the solution (4.16) is periodic of the frequencies ω and 3ω ($= \omega_{20}$). The higher-harmonic entrainment of this type was discussed in Sec. 4.4(b). In the second case, the solution (4.16) is periodic of the frequencies ω, $\omega/3$ ($= \omega_{10}$), and 3ω ($= \omega_{20}$). In the same manner as before the amplitudes r_{10} and r_{20} are given by solving

$$(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)^2 + \sigma_1^2 = A_1^2 r_{10}^2$$

$$[(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)^2 + \sigma_2^2] r_{20}^2 = \frac{1}{9} A_1^6$$

where

$$\sigma_1 = \frac{8n_1(k_2 - k_1)\omega_{11}}{\omega_1^2 k_2} = \frac{8n_1(k_2 - k_1)}{\mu\omega_1^2 k_2} (\omega_1 - \omega/3) \tag{5.70}$$

$$\sigma_2 = \frac{8n_1(k_1 - k_2)\ 21}{\omega_2^2 k_1} = \frac{8n_1(k_1 - k_2)}{\mu\omega_1^2 k_2} (\omega_1 - 3\omega)$$

The phase angles θ_{10} and θ_{20} are given by

$$\sin 3\theta_{10} = -\frac{\sigma_1}{A_1 r_{10}}$$
$$\cos 3\theta_{10} = \frac{1}{A_1 r_{10}}(\rho_1 - 2A_1^2 - r_{10}^2 - 2r_{20}^2)$$
$$\sin \theta_{20} = -\frac{3\sigma_2 r_{20}}{A_1^3} \qquad (5.71)$$
$$\cos \theta_{20} = \frac{3r_{20}}{A_1^3}(\rho_2 - 2A_1^2 - 2r_{10}^2 - r_{20}^2)$$

The stability of the steady-state solutions is tested as before.

(h) Entrained Oscillations And Their Frequency Components

By using the results in this section, the relationships between the types of the internal and external resonances, and the frequency components contained in the steady-state solutions are summarized in Table 5.2. The steady state (4), i.e., $r_{10} \neq 0$, $r_{20} \neq 0$, in the table corresponds to the entrained oscillations by the existence of the internal resonance $s\omega_{10} = \omega_{20}$ $(s \neq 1, 3)$.

5.5 Concluding Remarks

The internal resonance in a self-oscillatory system with a forcing term is investigated by making use of the averaging method. When the ratio between the two natural frequencies is in the neighborhood of a simple integer (different from unity) or a fraction, both the two natural frequencies are entrained by the frequencies which are integral multiples or submultiples of the driving frequency. The entrained oscillations are characterized by their waveforms which contain higher-harmonic or subharmonic components besides the harmonic component. The amplitude and phase characteristics of some representative cases are shown.

Table 5.2 The frequency components contained in the steady-state solutions (internal resonance : $s\omega_{10} = \omega_{20}$, where $s \neq 1$ or 3)

Type	is in the neighborhood of	$s = \frac{\omega_{20}}{\omega_{10}}$	Steady states			
			(1) $r_{10} = 0$, $r_{20} = 0$	(2) $r_{10} \neq 0$, $r_{20} = 0$	(3) $r_{10} = 0$, $r_{20} \neq 0$	(4) $r_{10} \neq 0$, $r_{20} \neq 0$
C1	$\frac{1}{2}\omega_1 = \frac{1}{3}\omega_2$	$\frac{3}{2}$	——	——	ω, 3ω	ω, 2ω, 3ω
C2	$\frac{1}{3}\omega_1 = \frac{1}{5}\omega_2$	$\frac{5}{3}$	——	——	——	ω, 3ω, 5ω
C3	$3\omega_1 = \frac{3}{2}\omega_2$	2	ω	ω, $\frac{1}{3}\omega$	——	ω, $\frac{1}{3}\omega$, $\frac{2}{3}\omega$
C4	$\frac{1}{3}\omega_1 = \frac{1}{7}\omega_2$	$\frac{7}{3}$	——	——	——	ω, 3ω, 7ω
C5	$3\omega_1 = \frac{3}{5}\omega_2$	5	ω	——	——	ω, $\frac{1}{3}\omega$, $\frac{5}{3}\omega$
C6	$3\omega_1 = \frac{3}{7}\omega_2$	7	ω	——	——	ω, $\frac{1}{3}\omega$, $\frac{7}{3}\omega$
C7	$3\omega_1 = \frac{1}{3}\omega_2$	9	——	——	ω, $\frac{1}{3}\omega$	ω, $\frac{1}{3}\omega$, 3ω

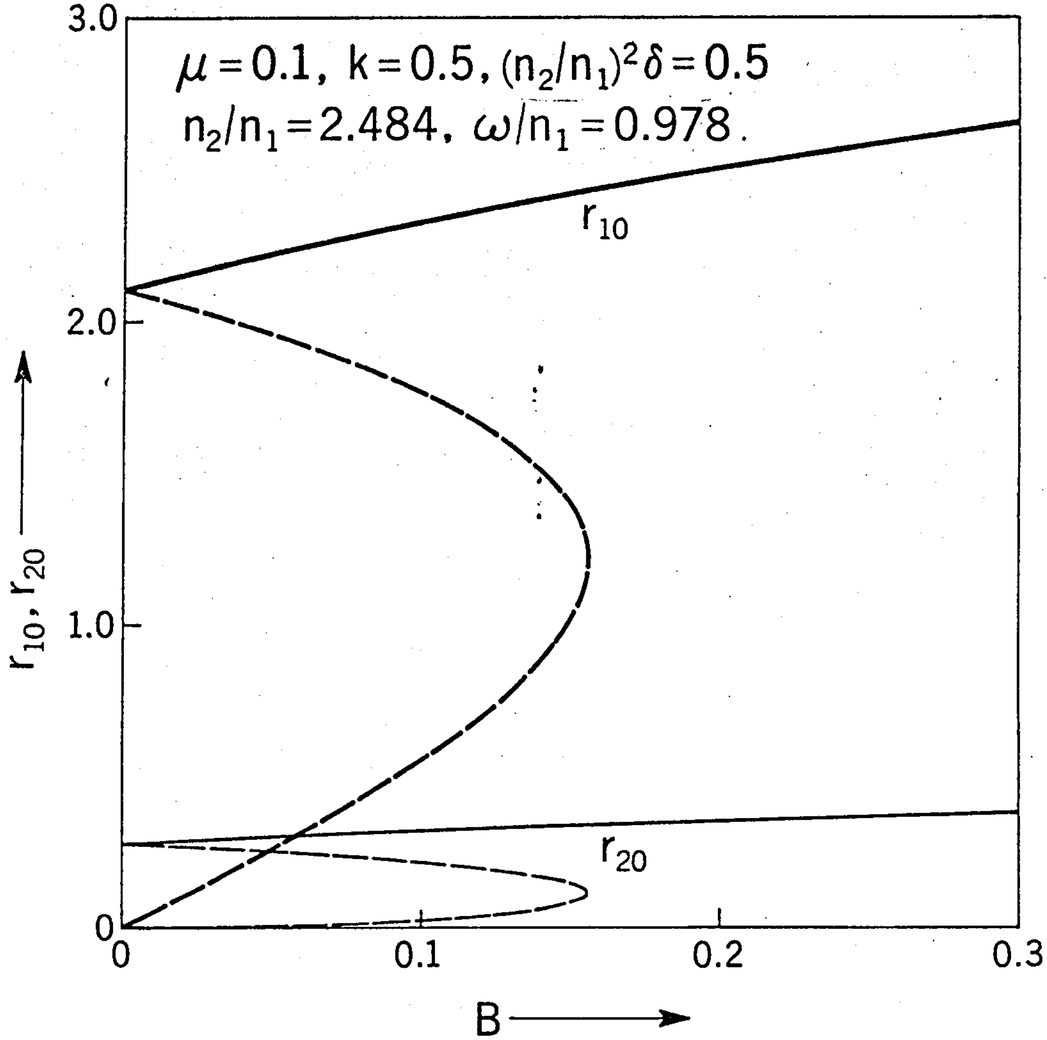


Fig. 5.1. Response curves with varying B ($\omega \cong \omega_1 \cong \omega_2/3$).

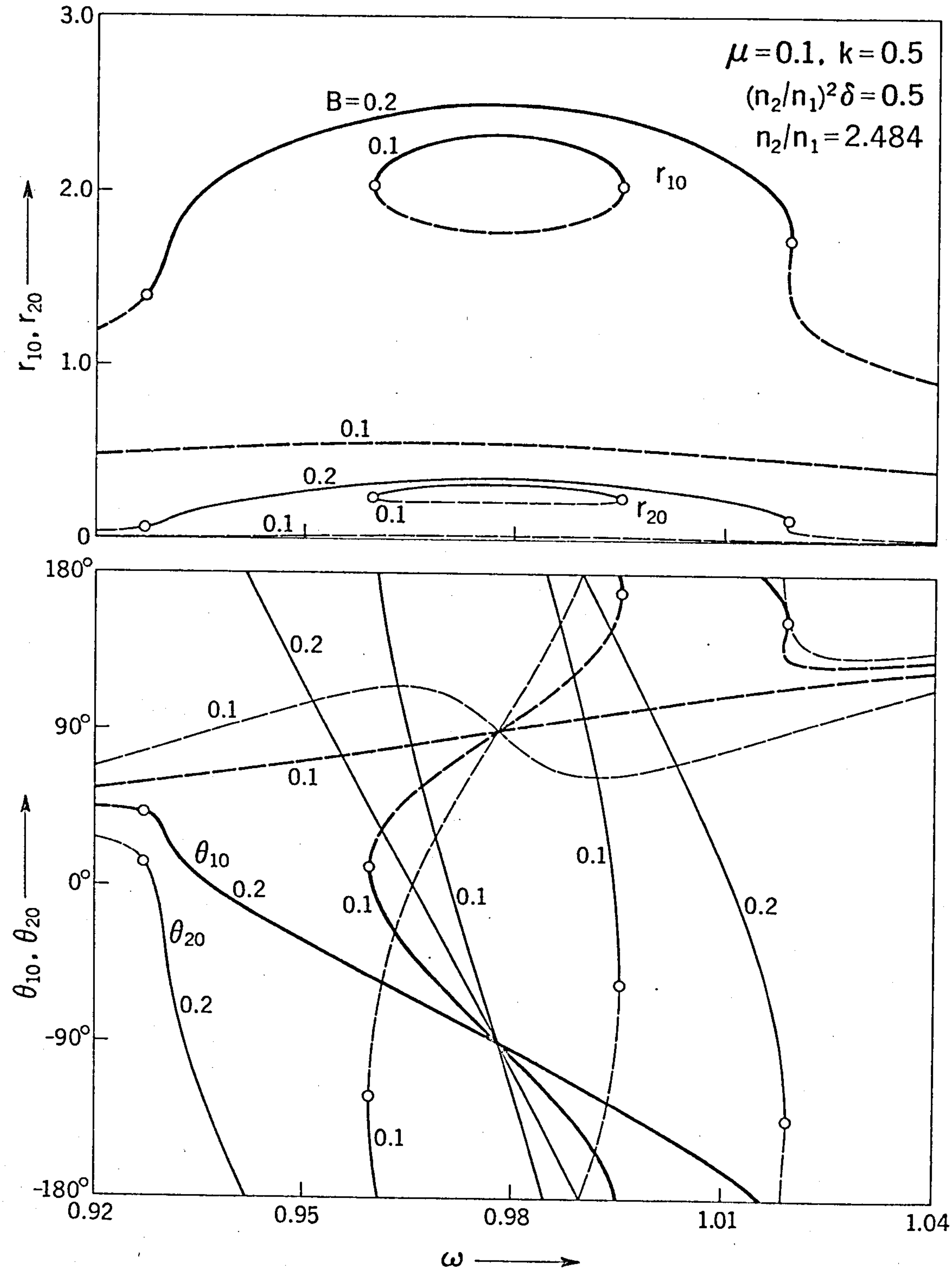


Fig. 5.2. Amplitude and phase characteristics of the entrained oscillation ($\omega \cong \omega_1 \cong \omega_2/3$).

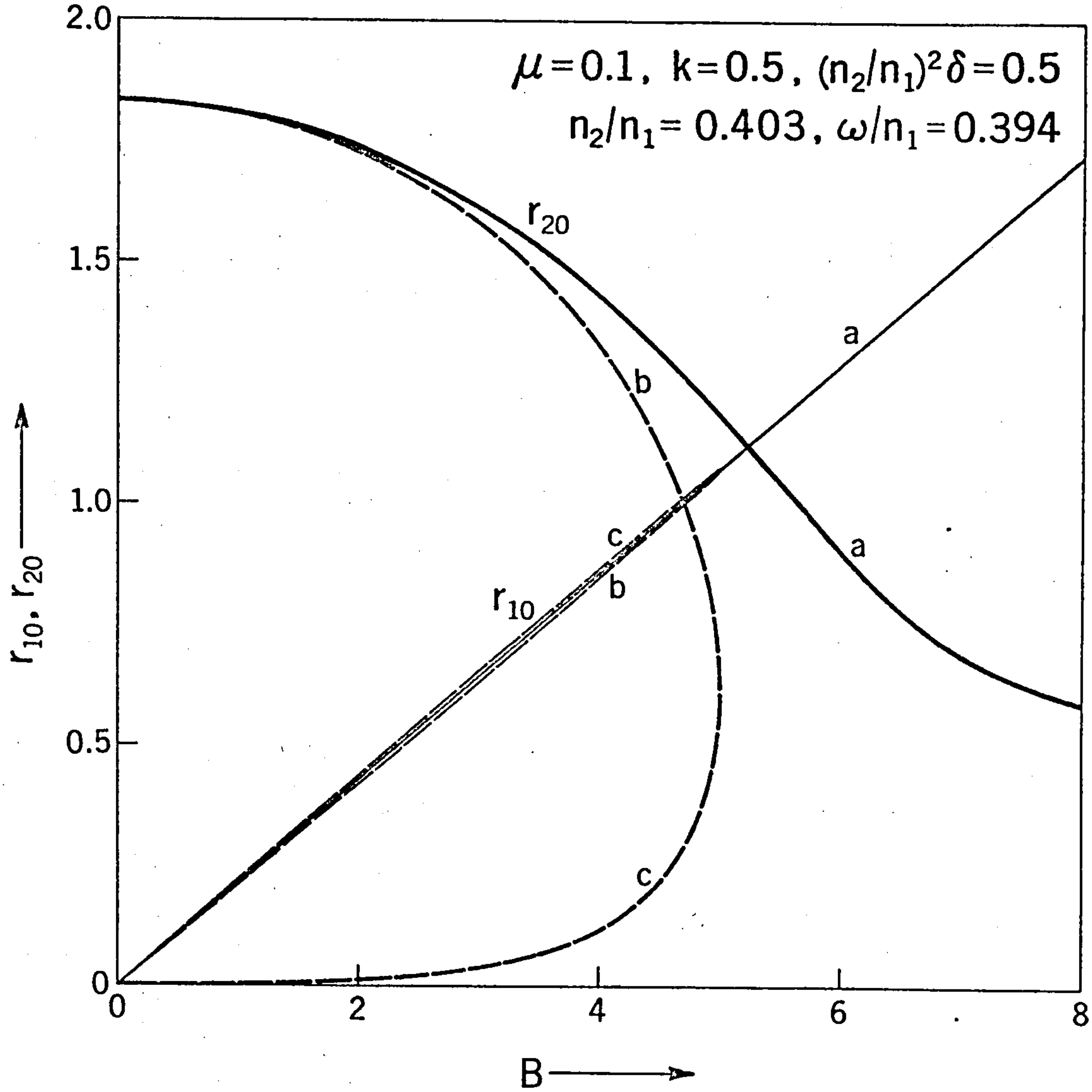


Fig. 5.3. Response curves with varying B ($\omega = \omega_1 = \omega_2/3$).

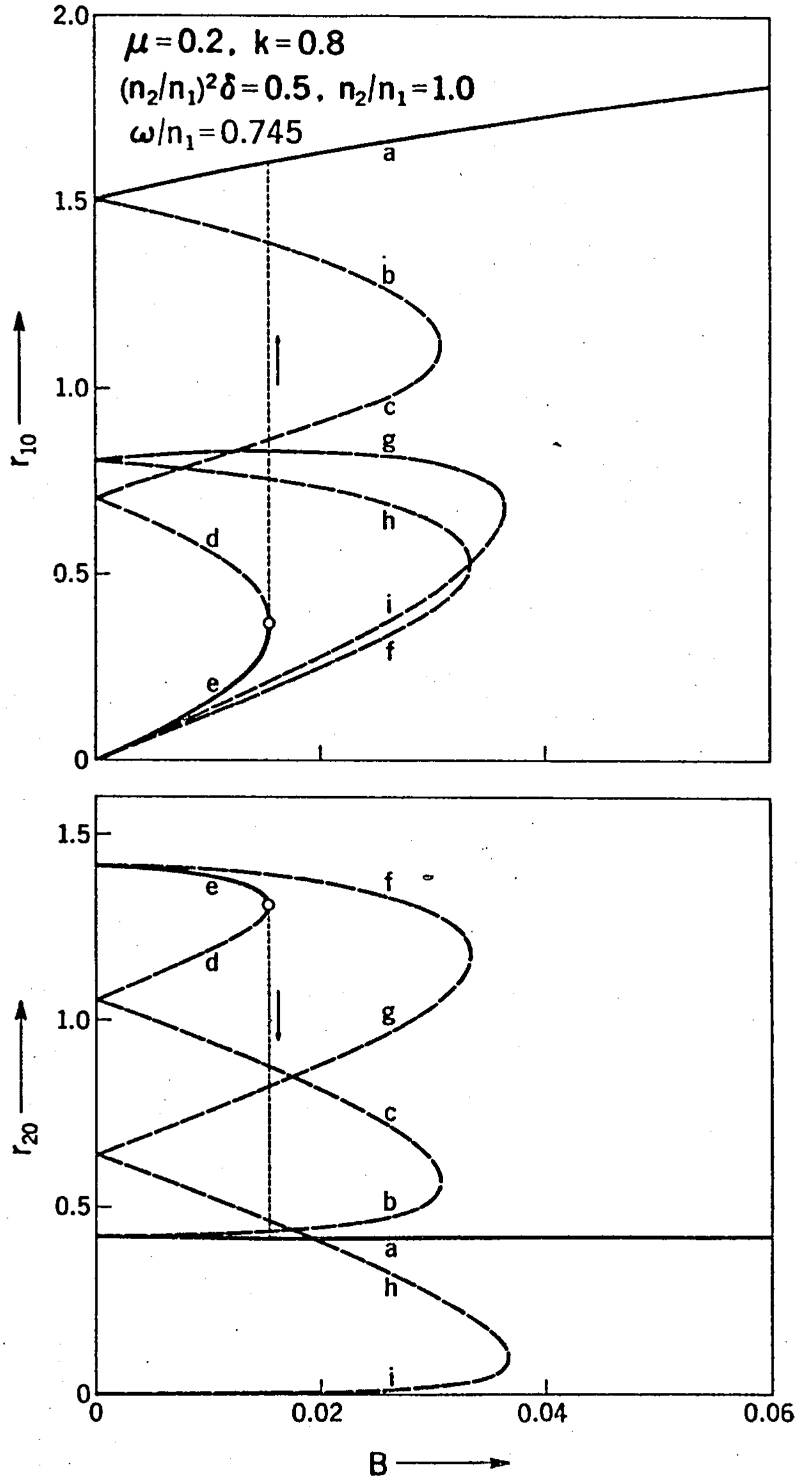


Fig. 5.4. Response curves with varying B ($\omega \cong \omega_1 \cong \omega_2/3$).

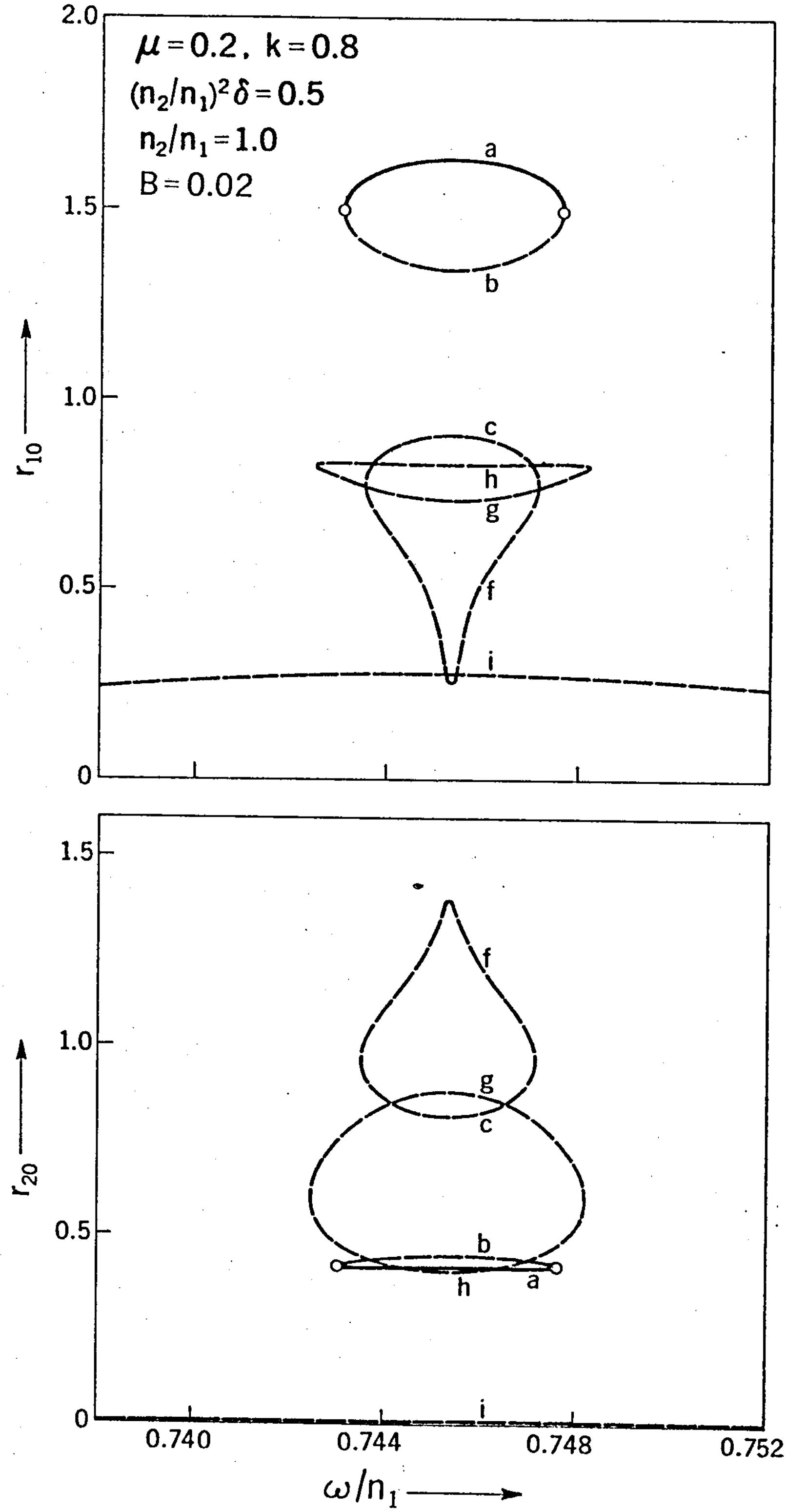


Fig. 5.5(a). Amplitude characteristic of the entrained oscillation ($\omega \cong \omega_1 \cong \omega_2/3$).

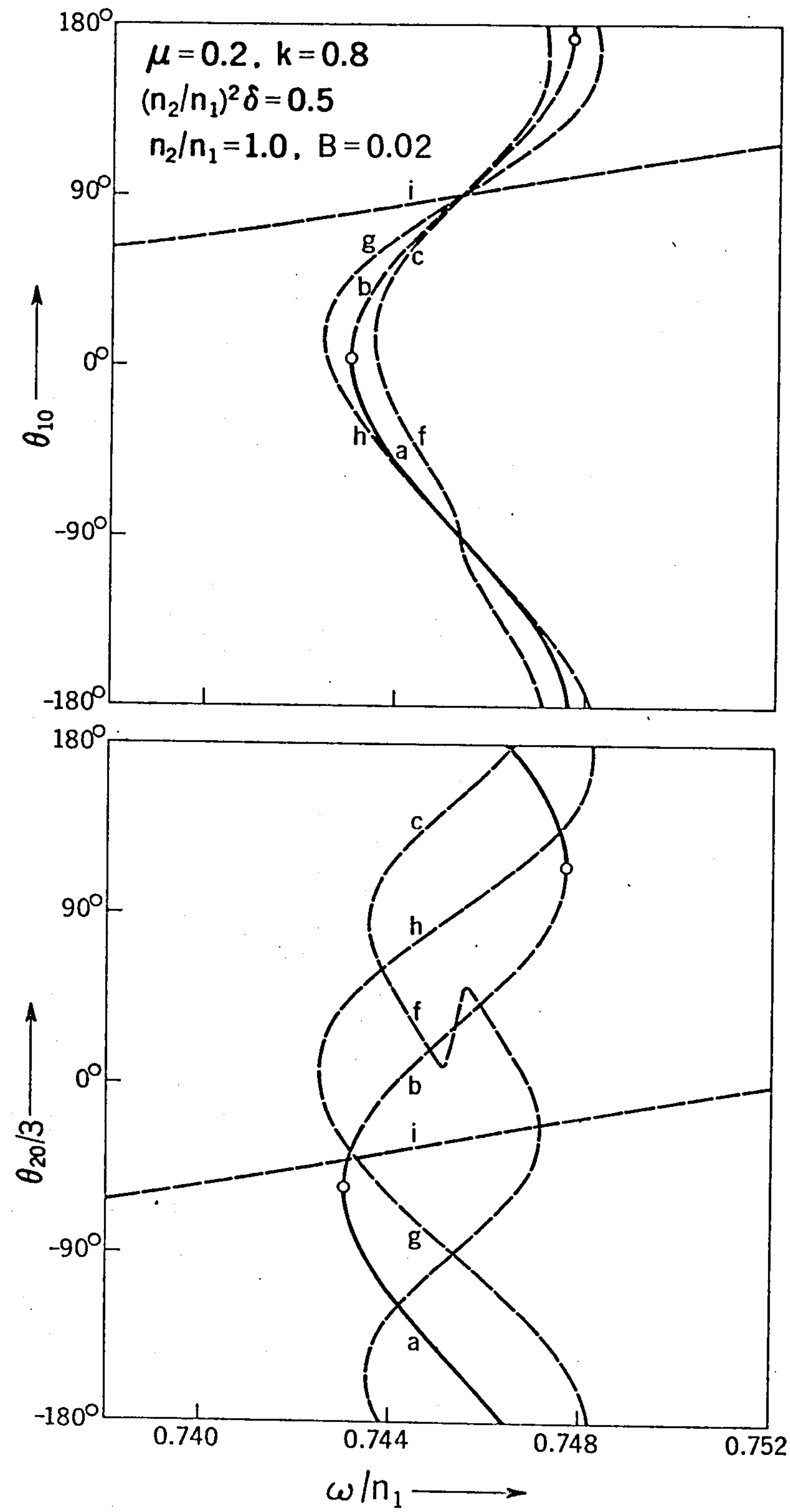


Fig. 5.5(b). Phase characteristic of the entrained oscillation ($\omega \simeq \omega_1 \simeq \omega_2/3$).

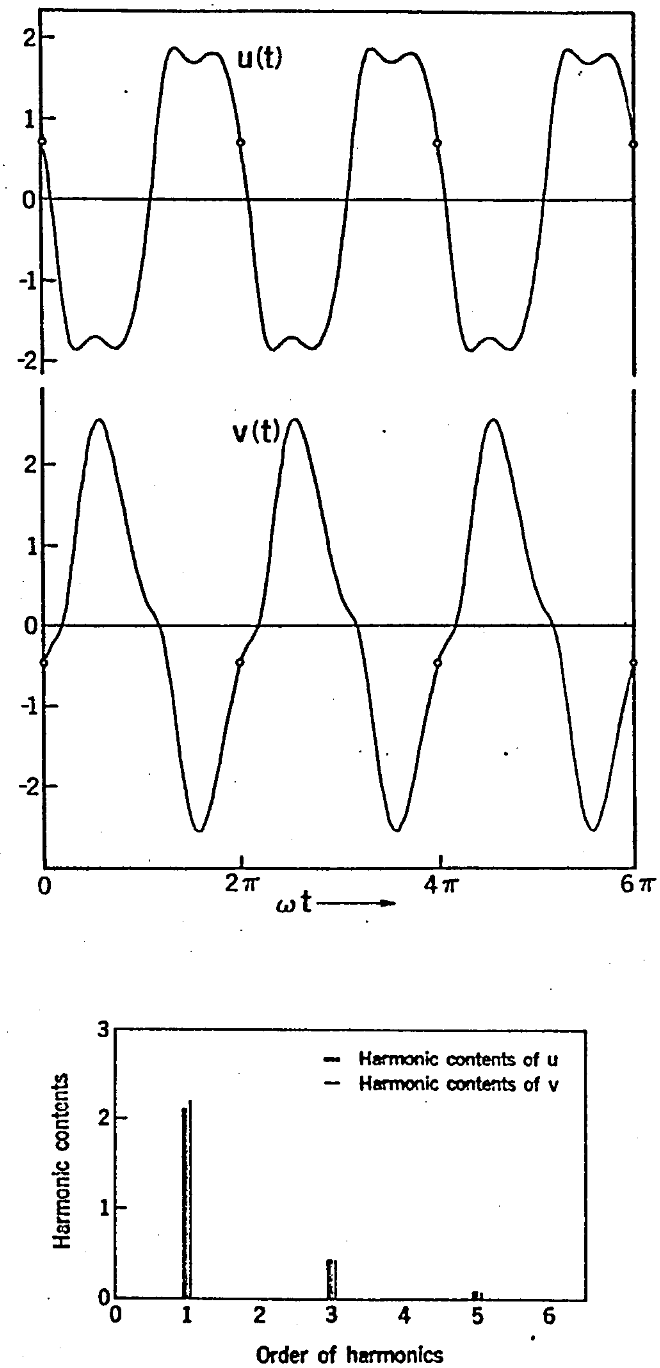


Fig. 5.6. Waveforms of the entrained oscillations and their harmonic analysis.[$\mu = 0.2$, $k = 0.8$, $(n_2/n_1)^2\delta = 0.5$, $n_2/n_1 = 1.0$, $\omega/n_1 = 0.745$, $B = 0.2$].

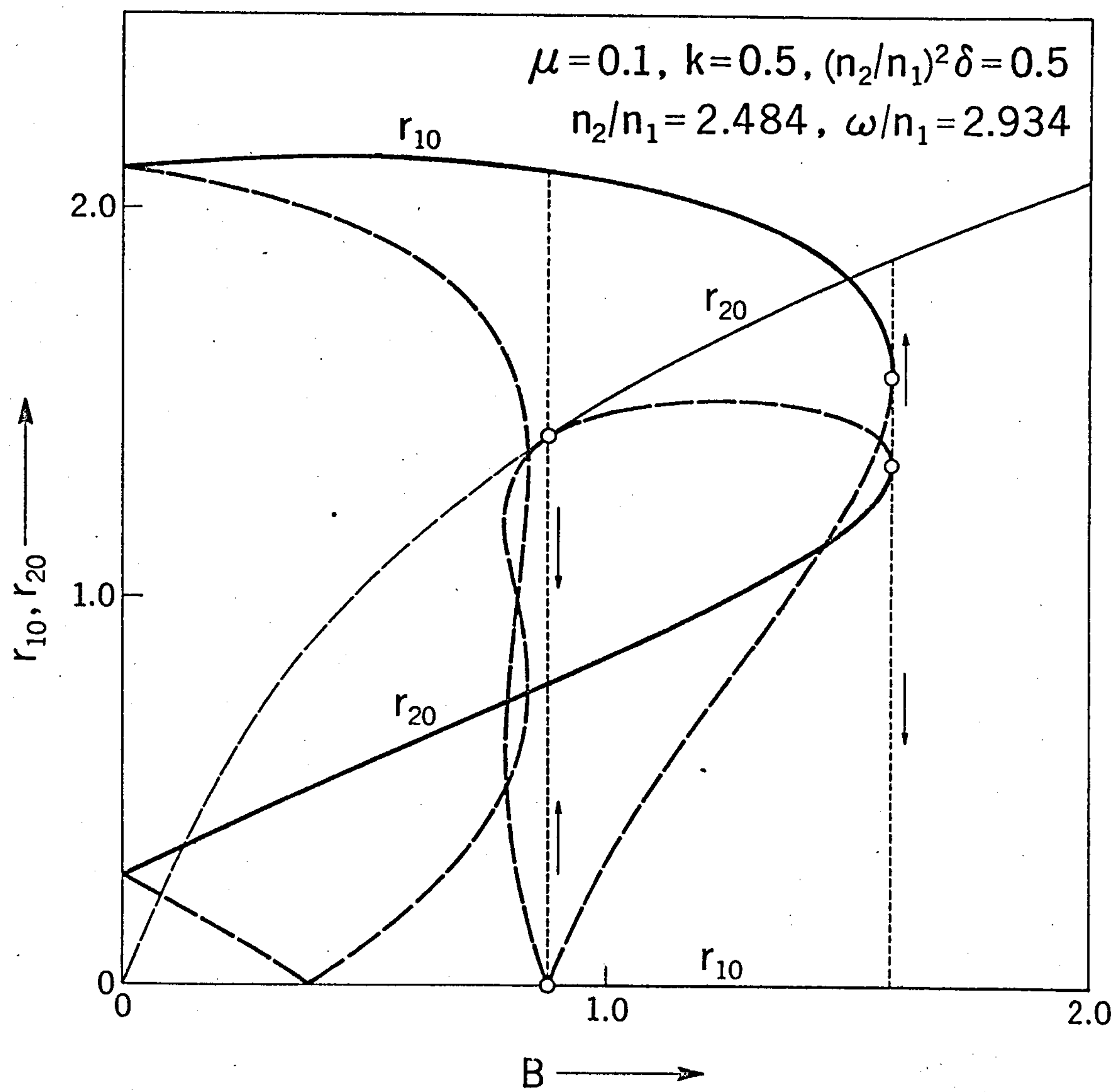


Fig. 5.7. Response curves with varying B ($\omega = 3\omega_1 = \omega_2$).

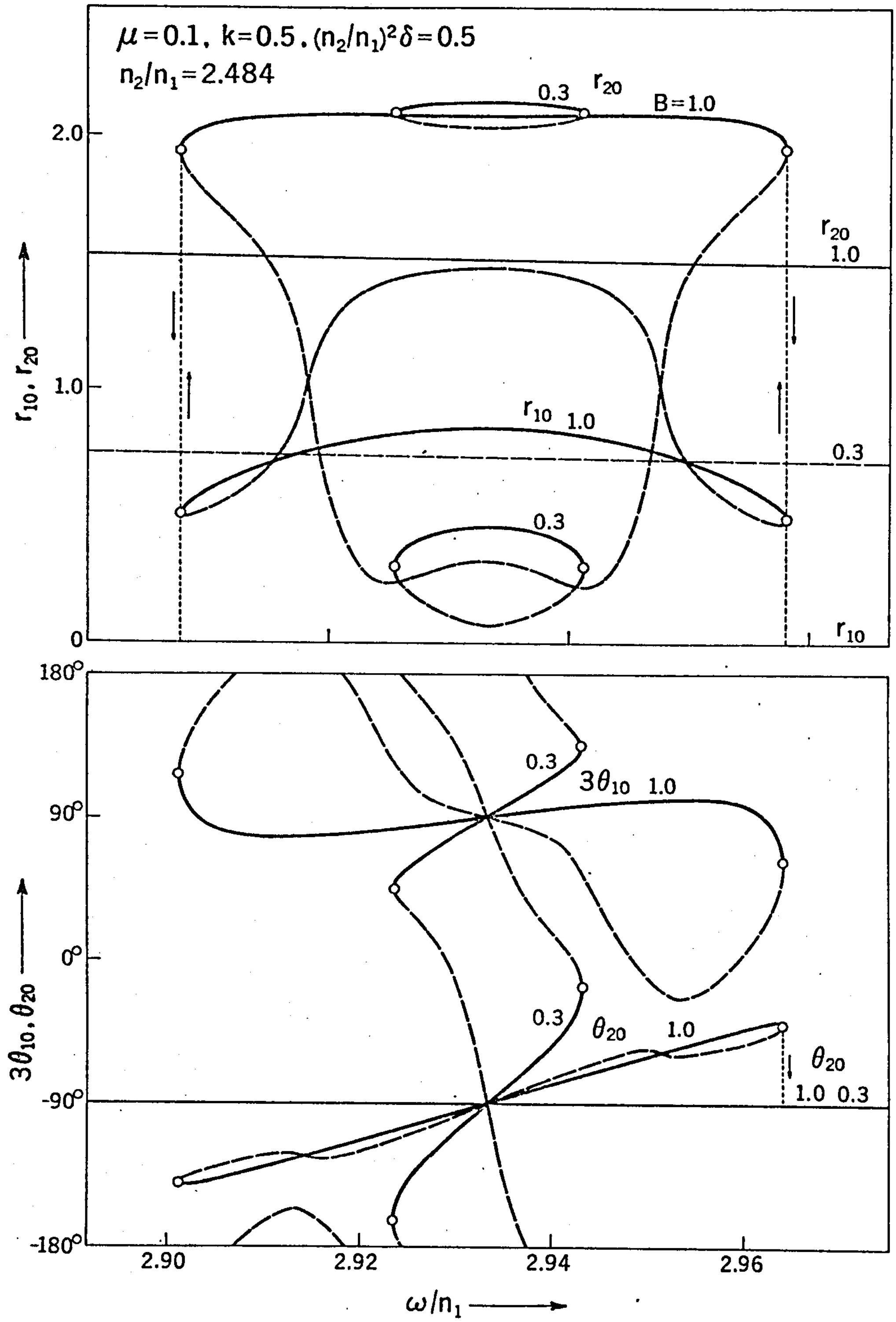


Fig. 5.8. Amplitude and phase characteristics of the entrained oscillation ($\omega \cong 3\omega_1 \cong \omega_2$).

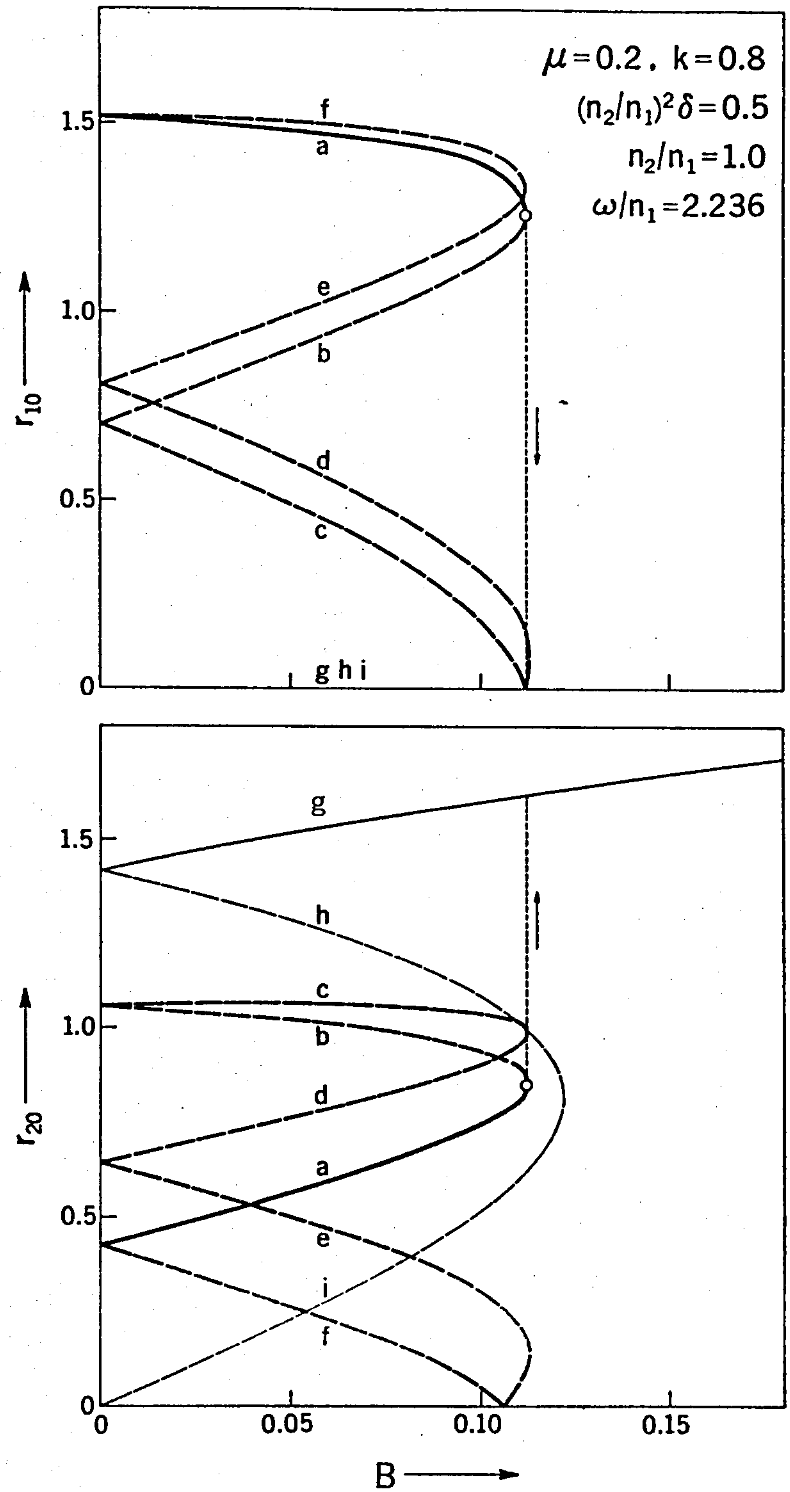


Fig. 5.9. Response curves with varying B ($\omega \cong 3\omega_1 \cong \omega_2$).

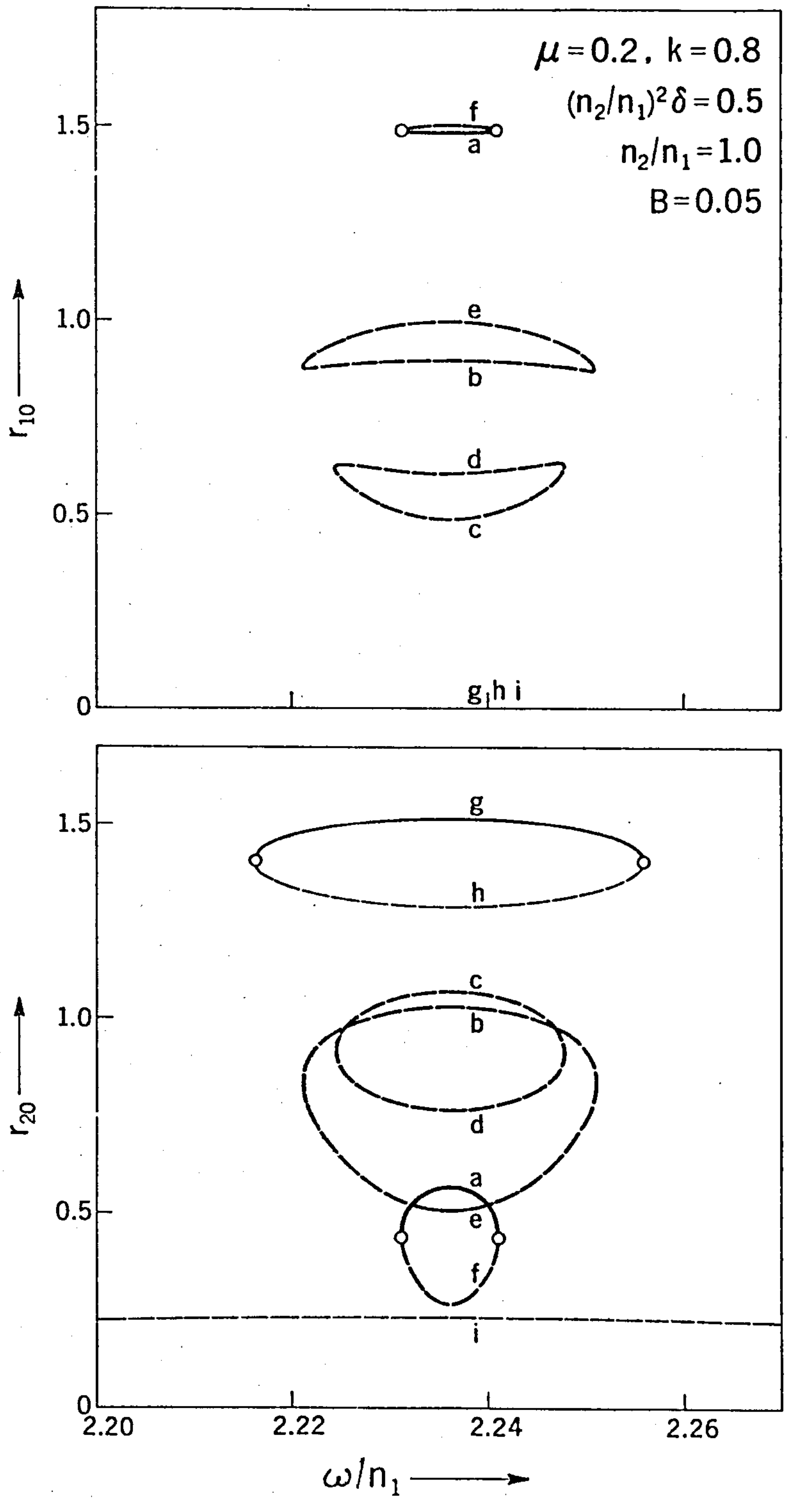


Fig. 5.10(a). Amplitude characteristic of the entrained oscillation ($\omega = 3\omega_1 = \omega_2$).

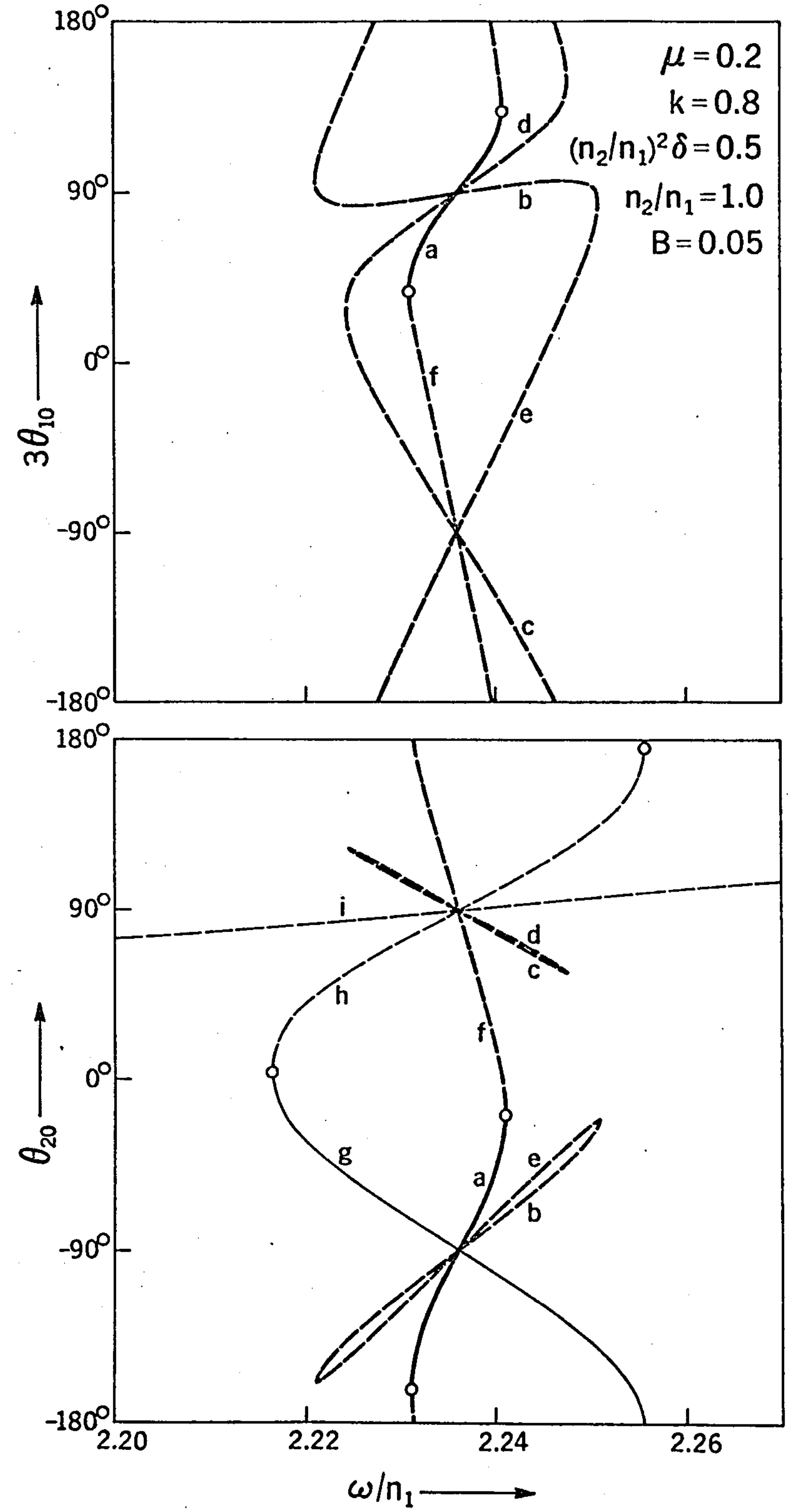
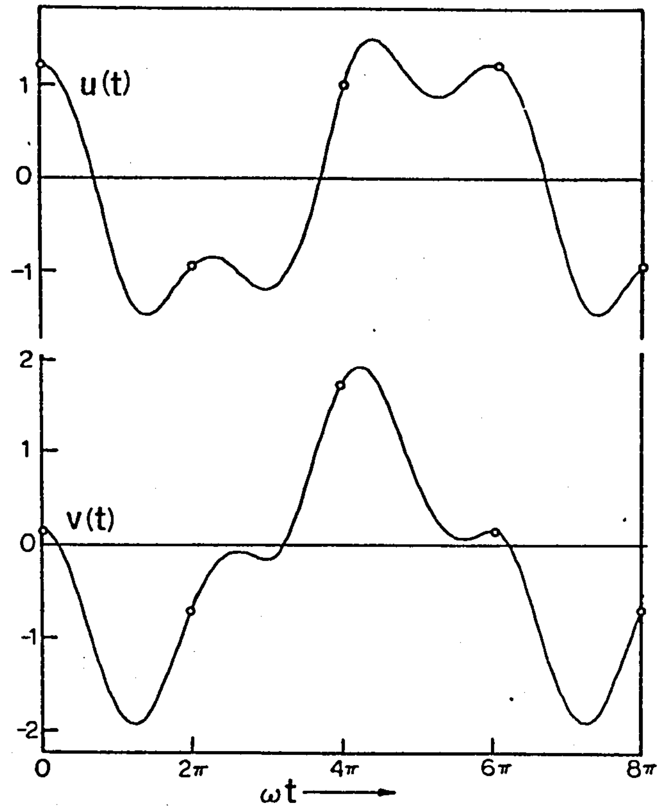


Fig. 5.10(b). Phase characteristic of the entrained oscillation ($\omega \cong 3\omega_1 \cong \omega_2$).



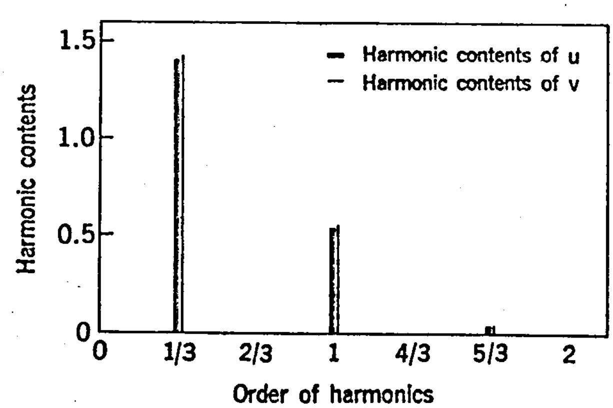


Fig. 5.11. Waveforms of the entrained oscillations and their harmonic analysis [$\mu = 0.2$, $k = 0.8$, $(n_2/n_1)^2\delta = 0.5$ $n_2/n_1 = 1.0$, $\omega/n_1 = 2.24$, $B = 0.06$].

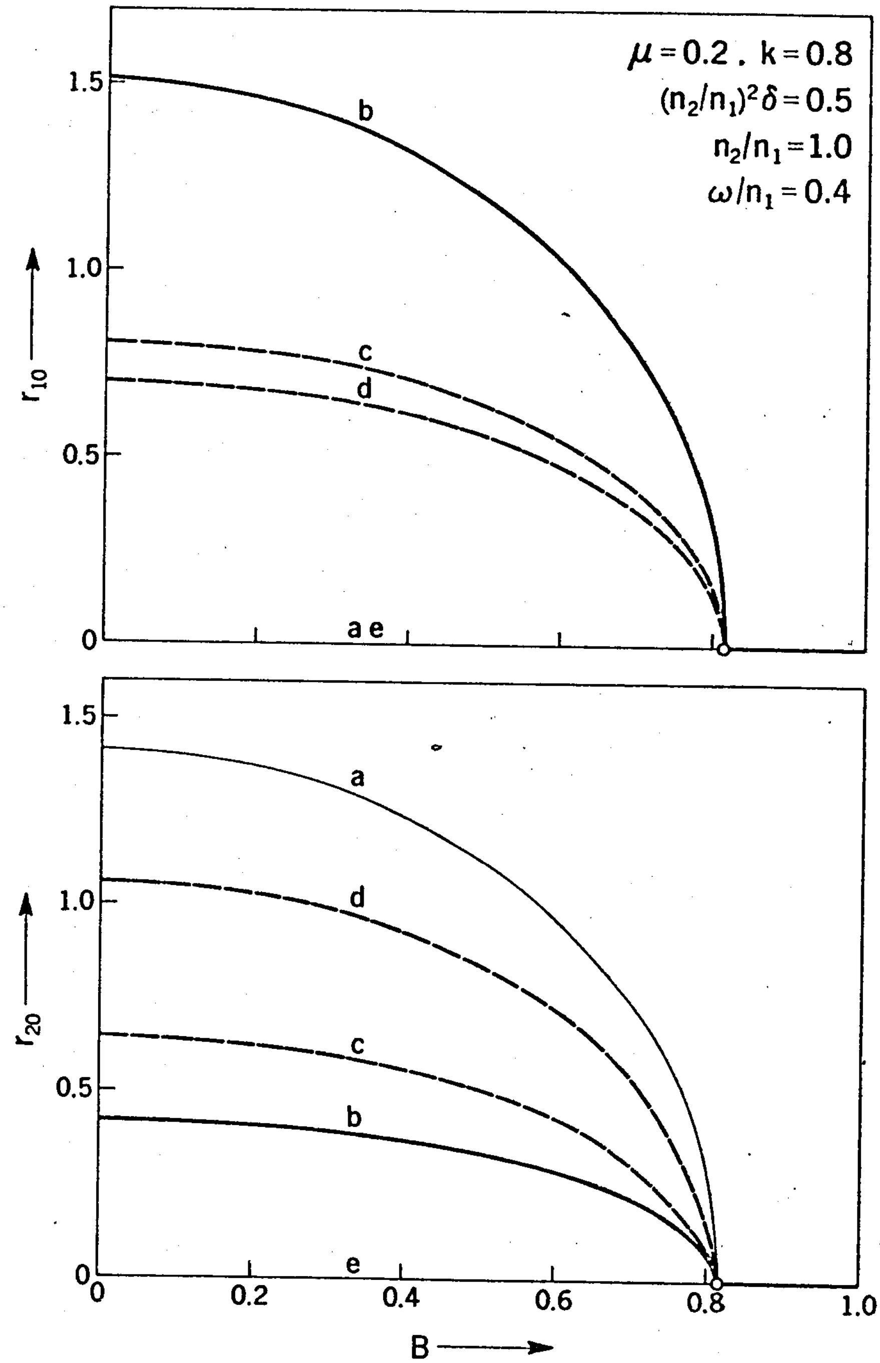


Fig. 5.12. Response curves with varying B (without external resonance).

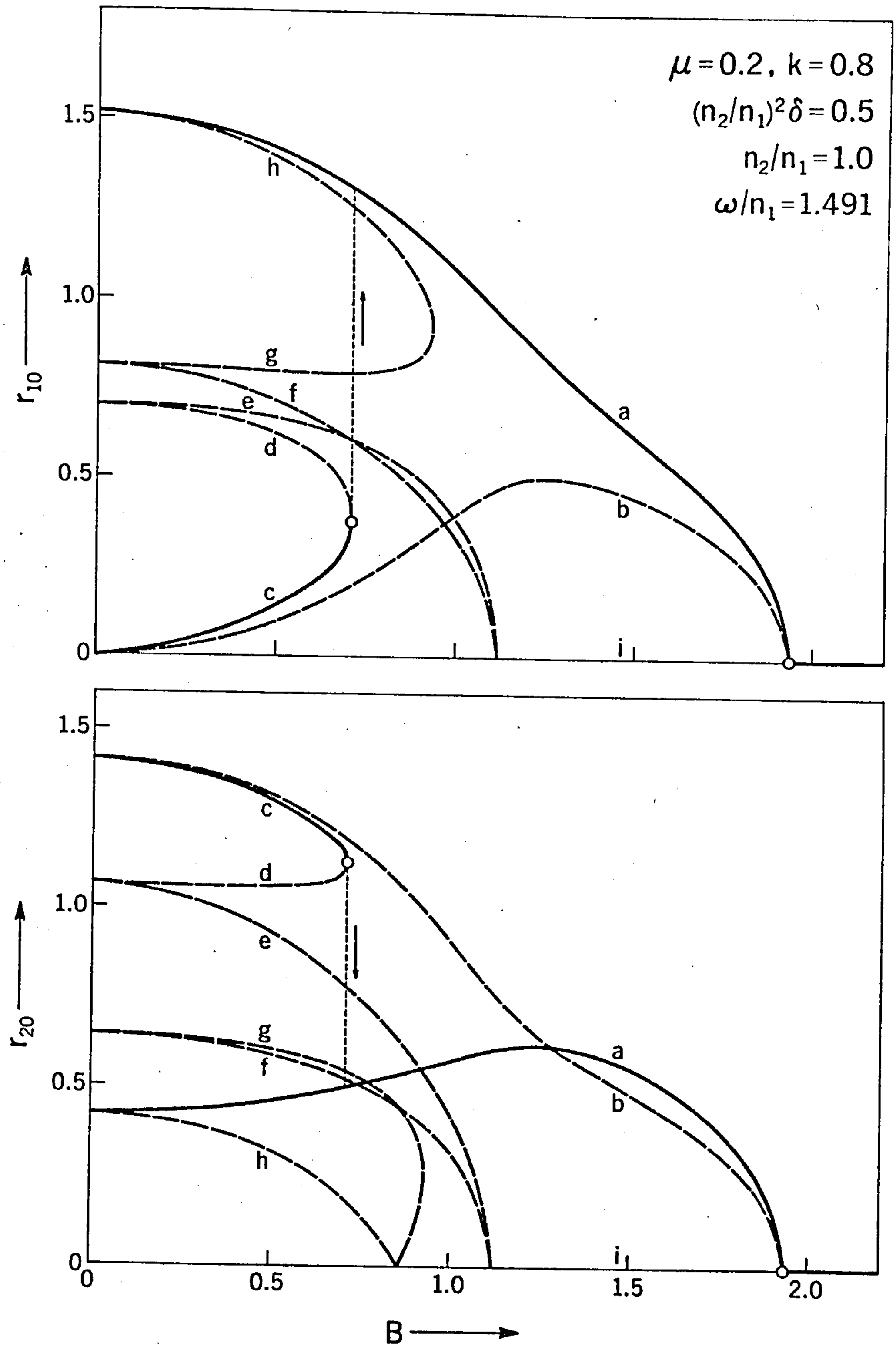


Fig. 5.13. Response curves with varying B ($\omega \cong 2\omega_1 \cong 2\omega_2/3$).

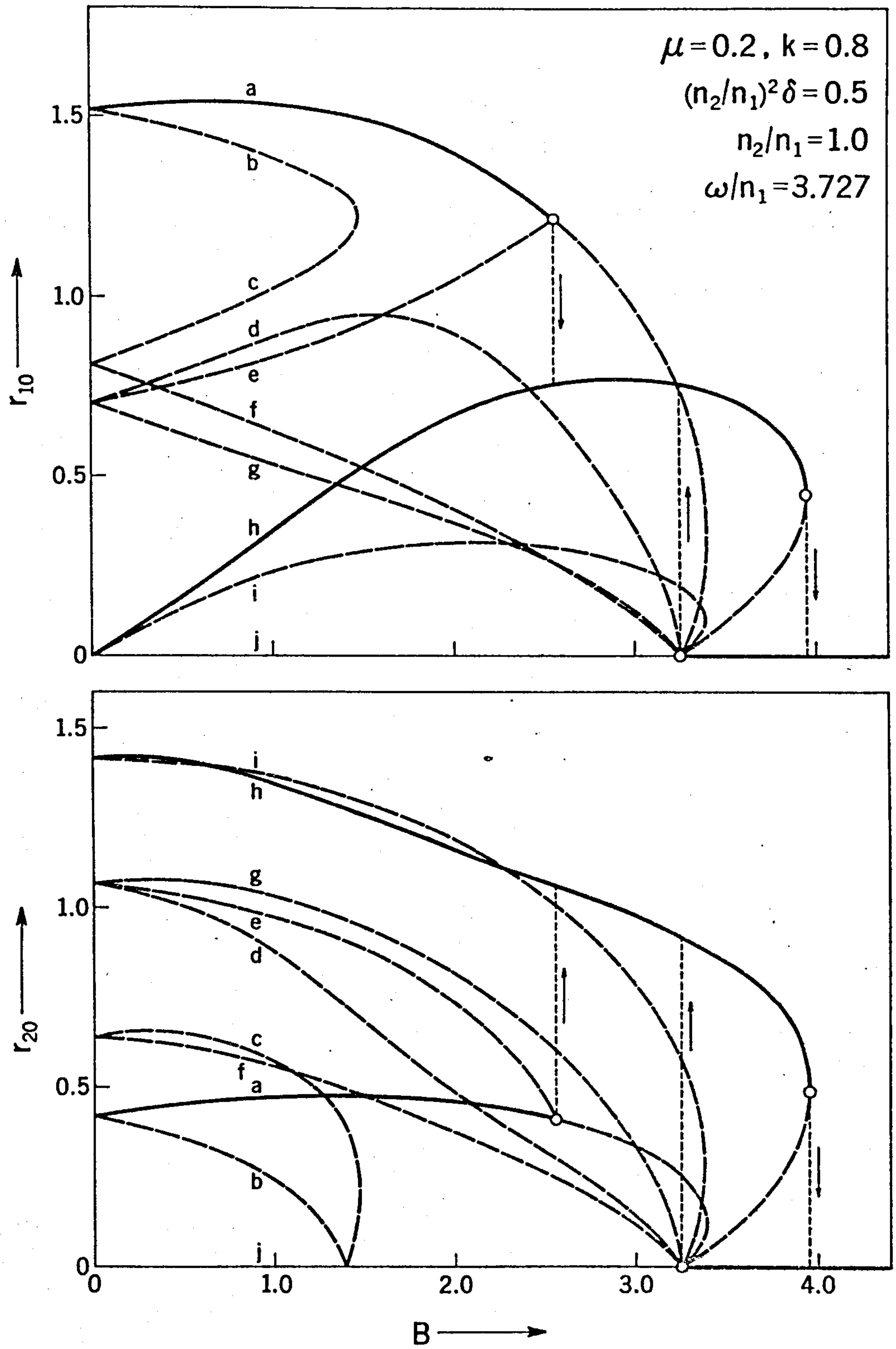


Fig. 5.14. Response curves with varying B ($\omega \cong 5\omega_1 \cong 5\omega_2/3$).

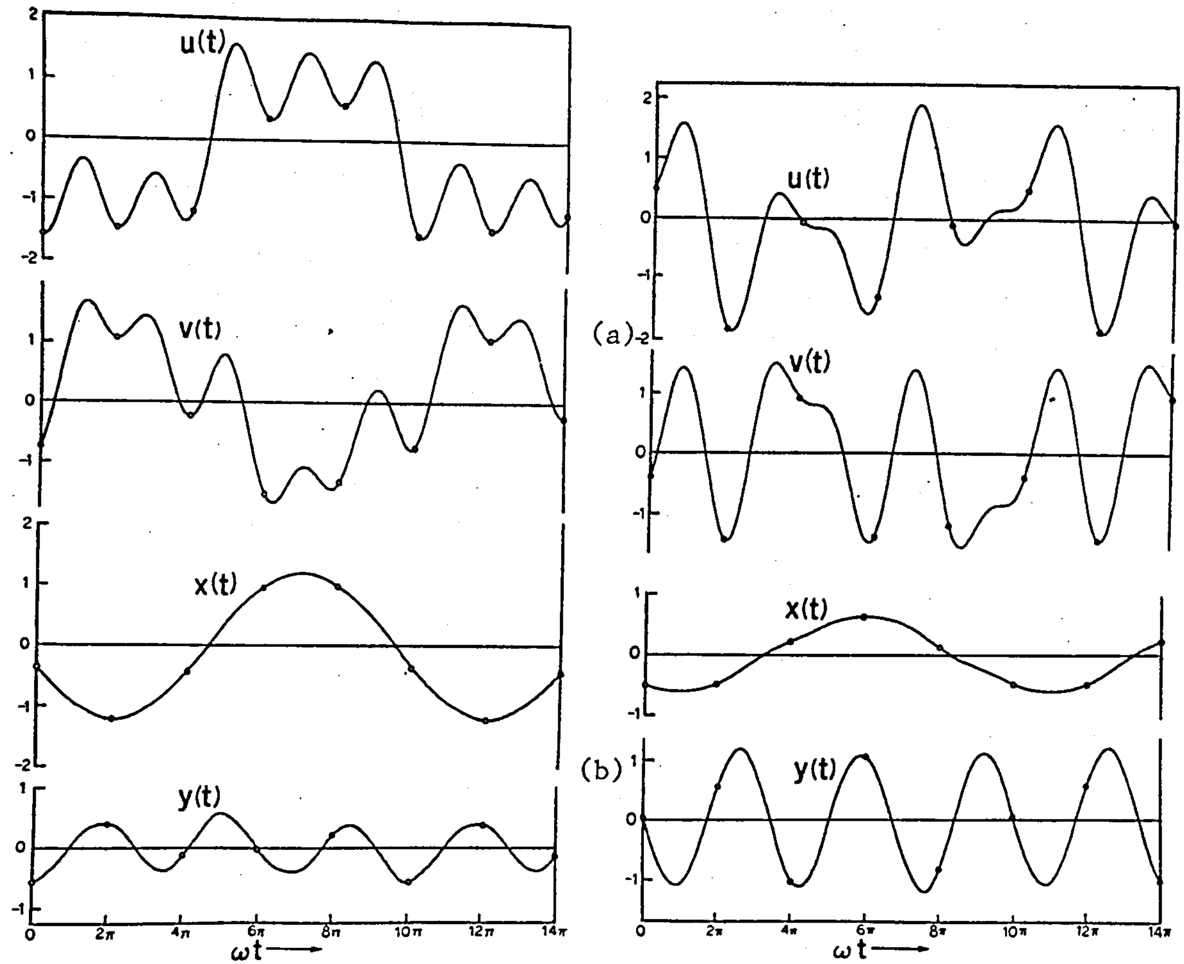


Fig. 5.15. Waveforms of two types of the entrained oscillations

[$\mu = 0.2$, $k = 0.8$, $(n_2/n_1)^2\delta = 0.5$, $n_2/n_1 = 1.0$,

$\omega/n_1 = 3.73$, $B = 2.0$].

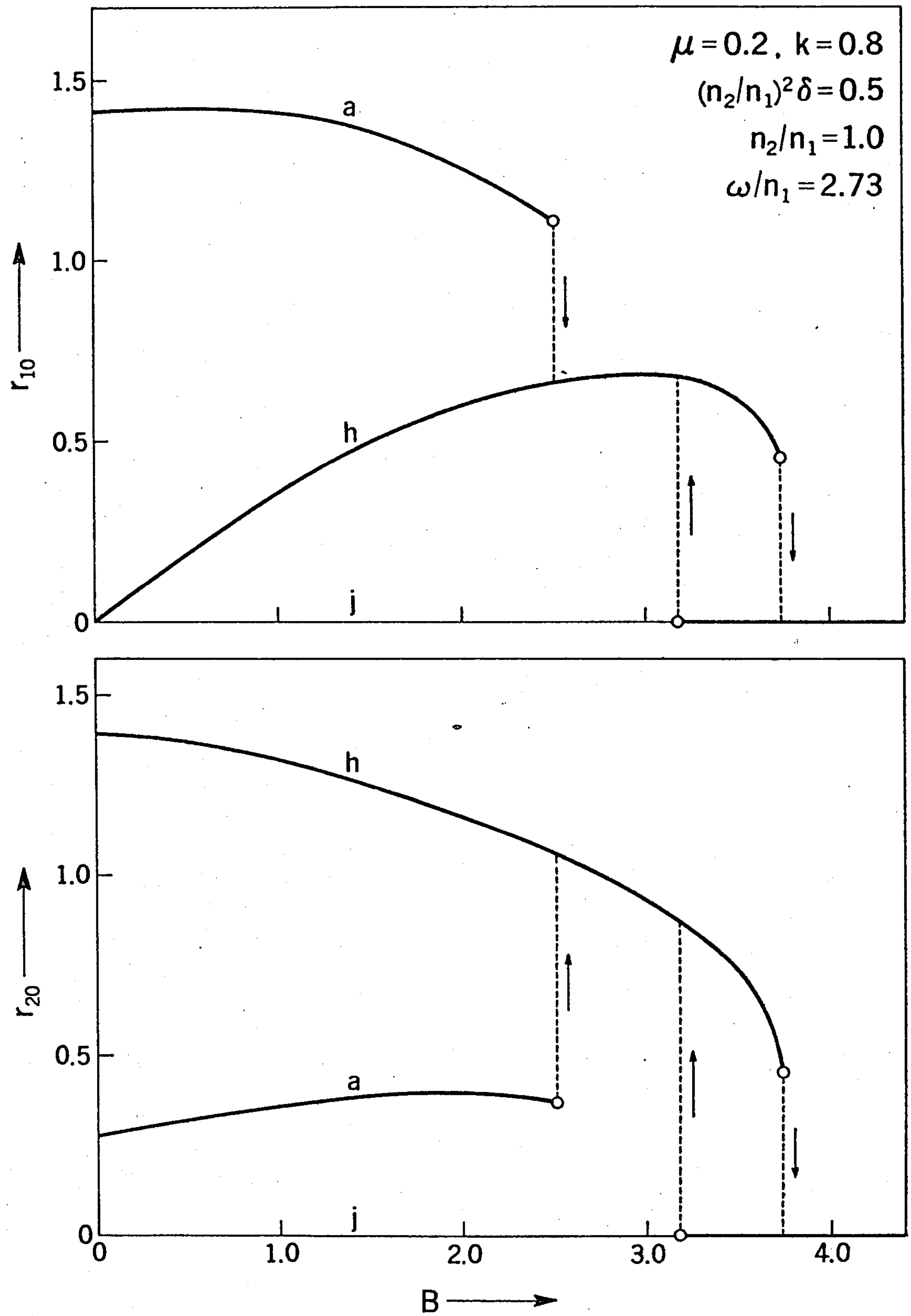


Fig. 5.16. Response curves obtained by analog computer analysis ($\omega \cong 5\omega_1 \cong 5\omega_2/3$).

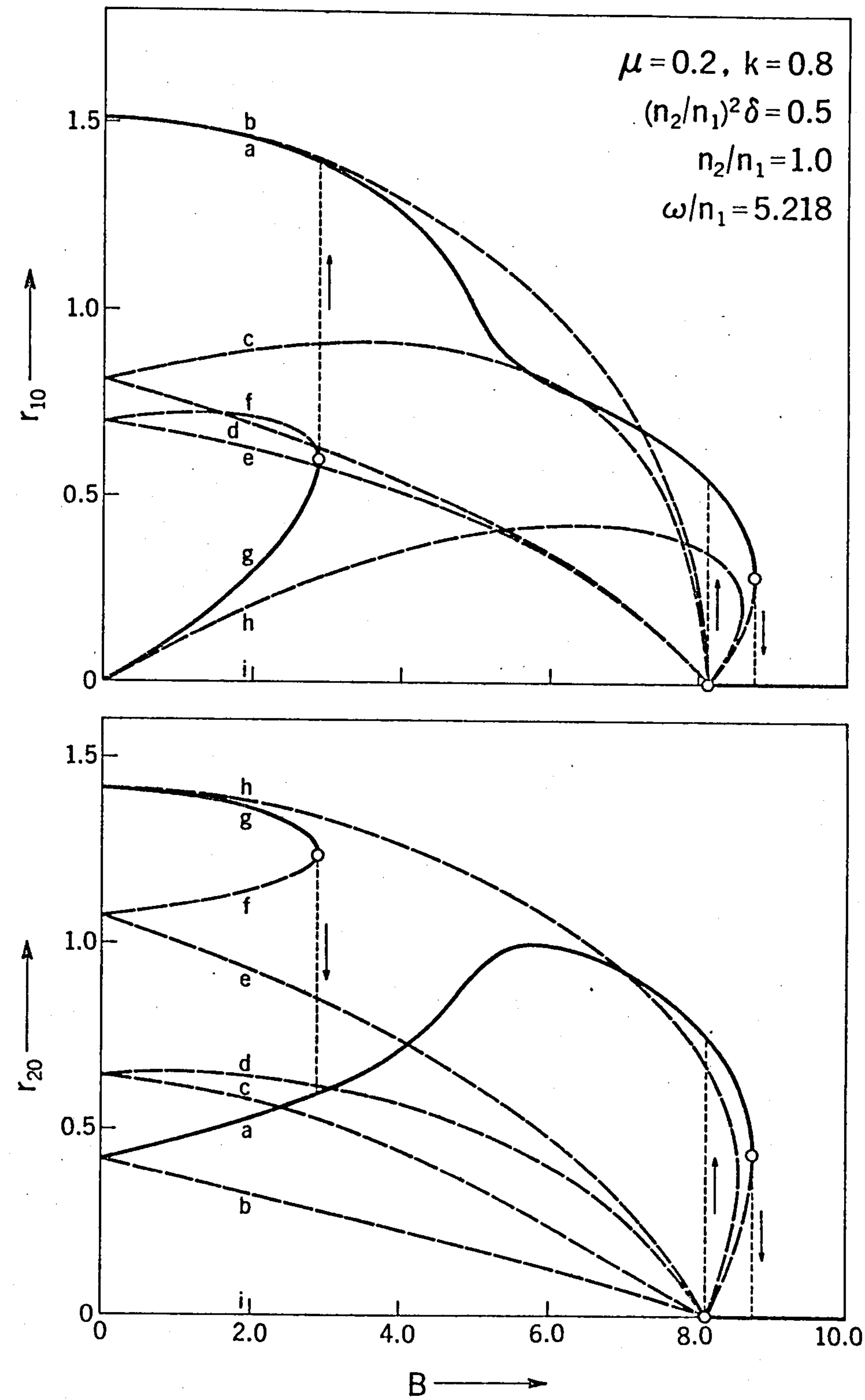


Fig. 5.17. Response curves with varying B ($\omega \cong 7\omega_1 \cong 7\omega_2/3$).

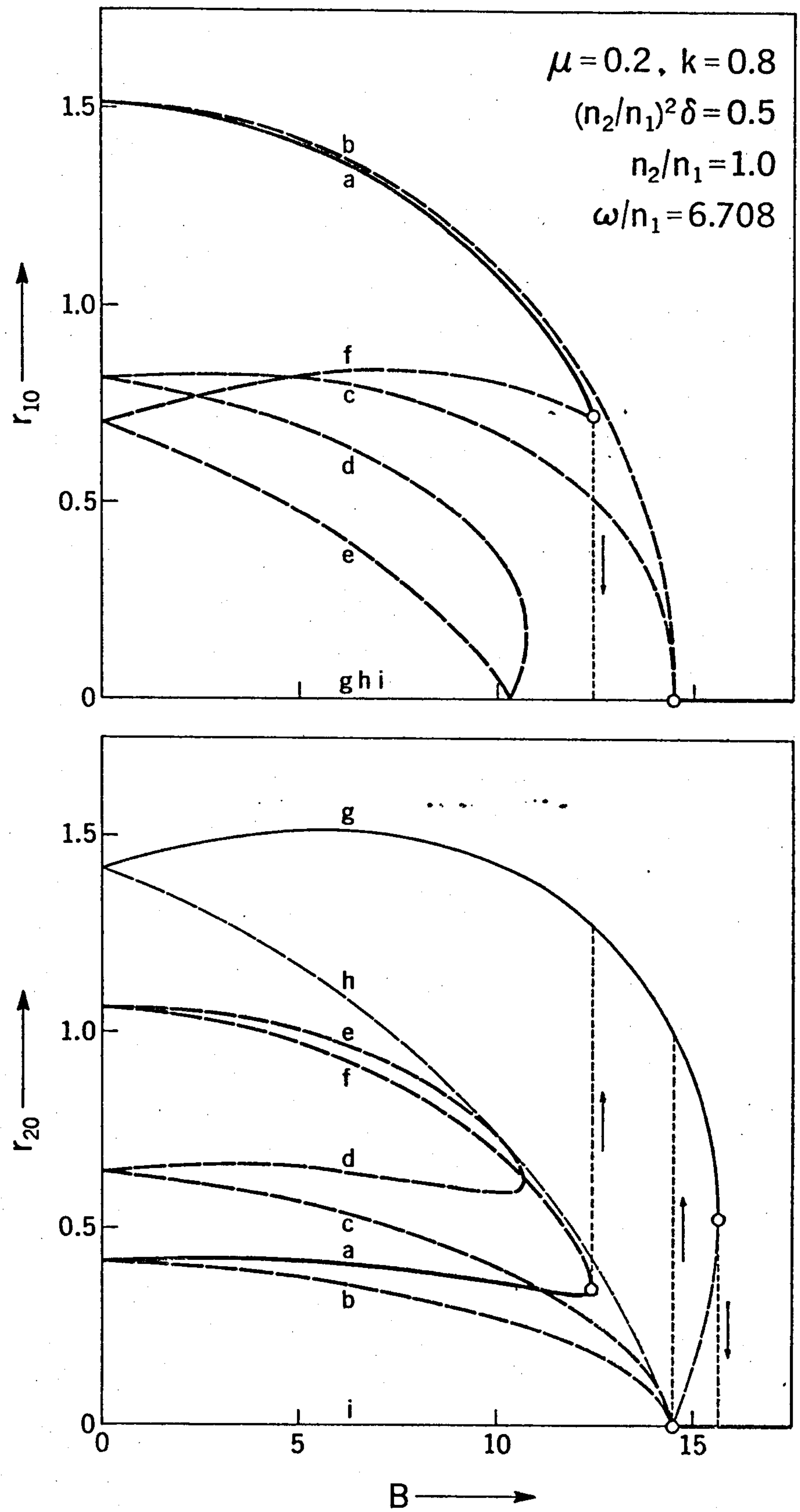


Fig. 5.18. Response curves with varying B ($\omega \cong 9\omega_1 \cong 3\omega_2$).

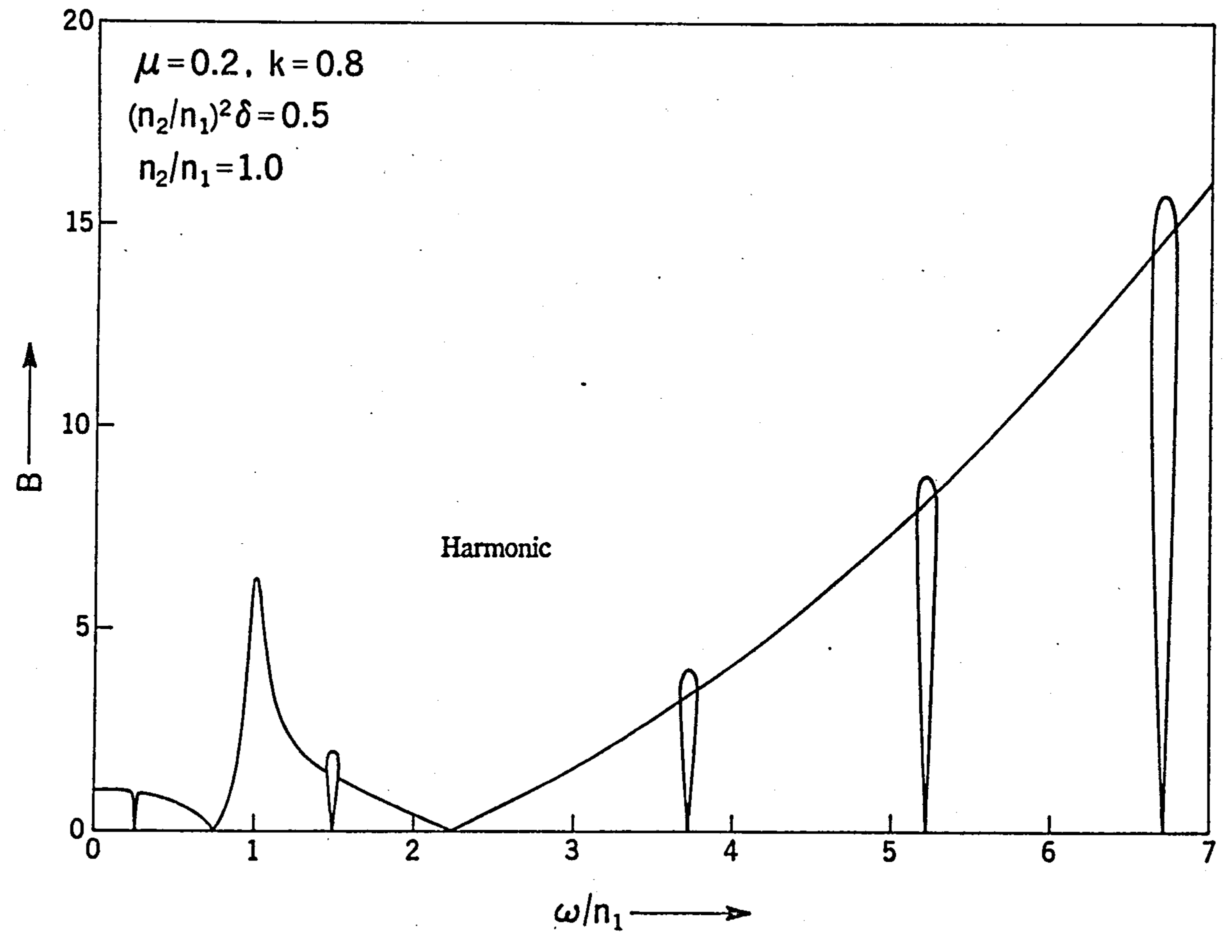


Fig. 5.19. Regions of frequency entrainment.

$\omega_1 : \omega_2 = 0.8 : 2.4$

$\omega = 1.5$ $\frac{1}{2}$ $\frac{3}{2}$

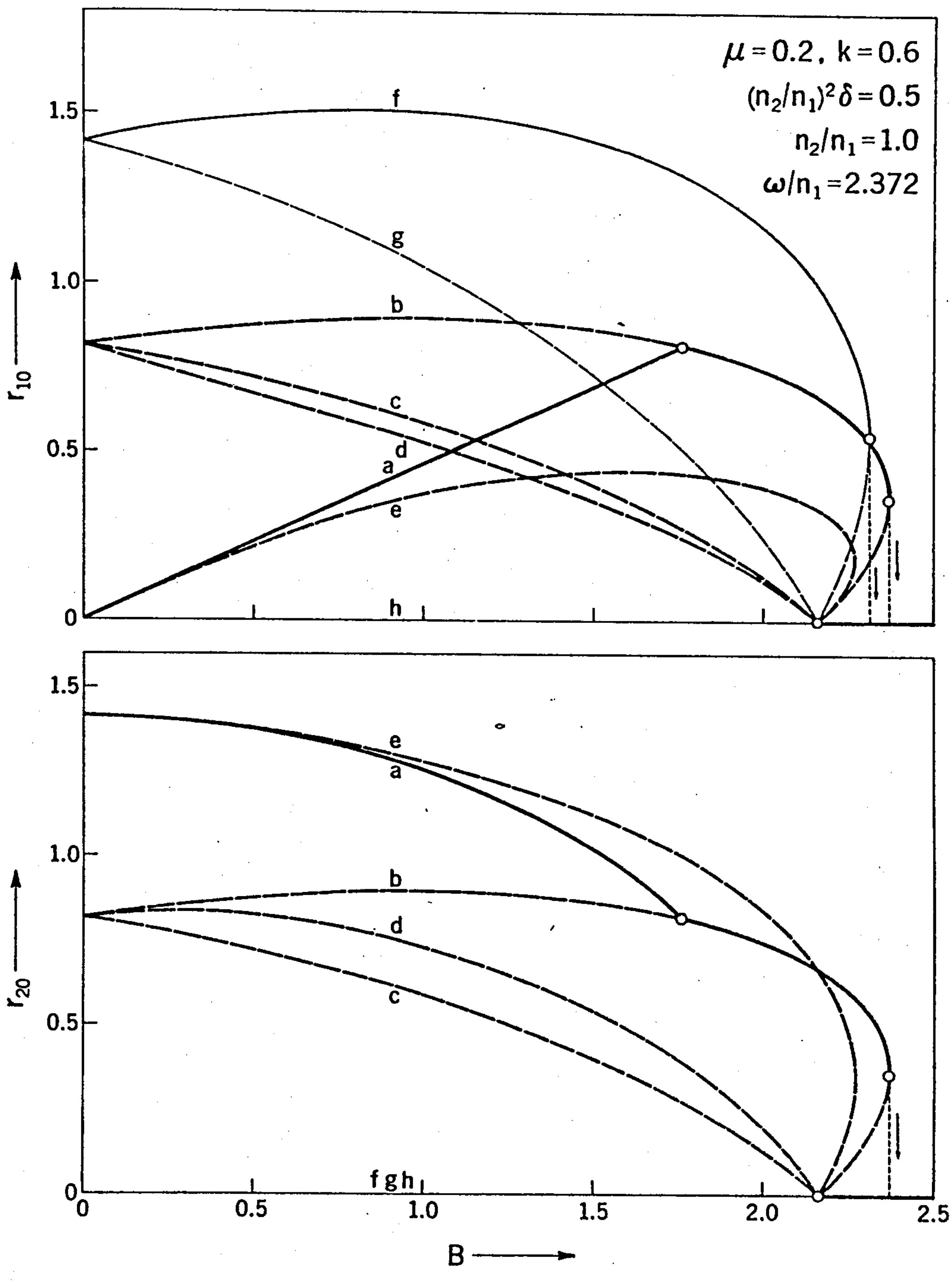


Fig. 5.20. Response curves with varying B ($\omega \cong 3\omega_1 \cong 3\omega_2/2$).

CHAPTER 6

INTERNAL RESONANCE UNDER SMALL COUPLING BETWEEN TWO RESONANT CIRCUITS

6.1 Introduction

The phenomenon of internal resonance in a self-oscillatory system was investigated in Chapter 3. If the two natural frequencies of the system are sufficiently close each other, the entrainment between these frequencies occurs owing to the nonlinearity of the system. The resulting oscillation is periodic one having a single harmonic component. This phenomenon occurs when the two resonant circuits whose resonant frequencies are close each other are coupled weakly. In the present chapter we discuss the behavior of such a self-oscillatory system under the influence of an external sinusoidal force. The frequency of a self-excited oscillation is entrained by the harmonic, higher-harmonic, and subharmonic frequencies of the external force. The amplitude and the phase characteristics of the entrained oscillations are derived by using the averaging method. The stability conditions are given by making use of the Routh-Hurwitz criterion.

6.2 Derivation of Autonomous Systems by Using the Averaging Method

If $n_1 \cong n_2$, $\chi_1 \cong 0$, and $\chi_2 \cong 0$ in Eqs. (1.7), then the two natural frequencies ω_1 and ω_2 are sufficiently close each other. In this case, as mentioned in Chap. 1, derivation of the standard form of Eqs. (1.21) and (1.25) ceases to be meaningful. Therefore, we consider the fundamental equations (1.7), i.e.,

$$\begin{aligned} \ddot{u} - \chi_1 \ddot{v} + n_1^2 u &= \mu n_1 (1 - u^2)\dot{u} + n_1^2 B \cos \omega t \\ \ddot{v} - \chi_2 \ddot{u} + n_2^2 v &= -\mu \frac{n_2^2}{n_1} \delta \dot{v} \end{aligned} \qquad (6.1)$$

Let the frequency of the self-excited oscillation be ω_0 which is in the neighborhood of both ω_1 and ω_2. As mentioned in Sec. 4.2, we consider the four cases according to the relationship between ω and ω_0 (see Table 4.4).

(a) Derivation of the Autonomous System When $\omega \cong \omega_0$

If the driving frequency ω is in the neighborhood of ω_0, we may expect that ω_0 is entrained by ω. Then the solution of Eqs. (6.1) is written as

$$\begin{aligned}
u(t) &= r_u(t) \cos [\omega t + \theta_u(t)] \\
v(t) &= r_v(t) \cos [\omega t + \theta_v(t)] \\
\dot{u}(t) &= -\omega r_u(t) \sin [\omega t + \theta_u(t)] \\
\dot{v}(t) &= -\omega r_v(t) \sin [\omega t + \theta_v(t)]
\end{aligned} \tag{6.2}$$

We assume that, for small values of μ, both the amplitudes $r_u(t)$, $r_v(t)$, and the phase angles $\theta_u(t)$, $\theta_v(t)$ are slowly varying functions of t. Substituting Eqs. (6.2) into Eqs. (6.1) and solving them for $\dot{r}_u$, $\dot{r}_v$, $\dot{\theta}_u$, and $\dot{\theta}_v$, we obtain

$$\begin{aligned}
\dot{r}_u &= f_4(r_u, r_v, \theta_u, \theta_v, t) \sin (\omega t + \theta_u) \\
\dot{r}_v &= g_4(r_u, r_v, \theta_u, \theta_v, t) \sin (\omega t + \theta_v) \\
r_u\dot{\theta}_u &= f_4(r_u, r_v, \theta_u, \theta_v, t) \cos (\omega t + \theta_u) \\
r_v\dot{\theta}_v &= g_4(r_u, r_v, \theta_u, \theta_v, t) \cos (\omega t + \theta_v)
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
f_4(r_u, r_v, \theta_u, \theta_v, t) = \frac{1}{\omega(1 - \chi_1\chi_2)}\Big\{&\mu\omega n_1(1 - \tfrac{1}{4} r_u^2)r_u \sin (\omega t + \theta_u) \\
&+ [n_1^2 - (1 - \chi_1\chi_2)\omega^2]r_u \cos (\omega t + \theta_u) - \mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_v \sin (\omega t + \theta_v) \\
&+ \chi_1 n_2^2 r_v \cos (\omega t + \theta_v) - \tfrac{1}{4}\mu\omega n_1 r_u^3 \sin (3\omega t + 3\theta_u) - n_1^2 B \cos \omega t\Big\}
\end{aligned}$$

$$g_4(r_u, r_v, \theta_u, \theta_v, t) = \frac{1}{\omega(1 - \chi_1\chi_2)}\Big\{\mu\omega\chi_2 n_1(1 - \frac{1}{4} r_u^2)r_u \sin(\omega t + \theta_u) \quad (6.4)$$

$$+ \chi_2 n_1^2 r_u \cos(\omega t + \theta_u) - \mu\omega\frac{n_2^2}{n_1}\delta r_v \sin(\omega t + \theta_v)$$

$$+ [n_2^2 - (1 - \chi_1\chi_2)\omega^2]r_v \cos(\omega t + \theta_v) - \frac{1}{4}\mu\omega\chi_2 n_1 r_u^3 \sin(3\omega t + 3\theta_u)$$

$$- \chi_2 n_1^2 B \cos \omega t\Big\}$$

Upon application of the averaging method, Eqs. (6.3) can be transformed into an autonomous system, i.e.,

$$\dot{r}_u = \lim_{T\to\infty}\frac{1}{T}\int_0^T f_4(r_u, r_v, \theta_u, \theta_v, t) \sin(\omega t + \theta_u)dt$$

$$= \frac{1}{2\omega(1 - \chi_1\chi_2)} [\mu\omega n_1(1 - \frac{1}{4} r_u^2)r_u + \chi_1 n_2^2 r_v \sin(\theta_u - \theta_v)$$

$$-\mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_v \cos(\theta_u - \theta_v) - n_1^2 B \sin\theta_u]$$

$$\dot{r}_v = \frac{1}{2\omega(1 - \chi_1\chi_2)} [-\mu\omega\frac{n_2^2}{n_1}\delta r_v - \chi_2 n_1^2 r_u \sin(\theta_u - \theta_v)$$

$$+\mu\omega\chi_2 n_1(1 - \frac{1}{4} r_u^2)r_u \cos(\theta_u - \theta_v) - \chi_2 n_1^2 B \sin\theta_v]$$

$$(6.5)$$

$$r_u\dot{\theta}_u = \frac{1}{2\omega(1 - \chi_1\chi_2)}\Big\{[n_1^2 - (1 - \chi_1\chi_2)\omega^2]r_u + \mu\omega\chi_1\frac{n_2^2}{n_1}\delta r_v \sin(\theta_u - \theta_v)$$

$$+ \chi_1 n_2^2 r_v \cos(\theta_u - \theta_v) - n_1^2 B \cos\theta_u\Big\}$$

$$r_v\dot{\theta}_v = \frac{1}{2\omega(1 - \chi_1\chi_2)}\Big\{[n_2^2 - (1 - \chi_1\chi_2)\ ^2]r_v + \mu\omega\chi_2 n_1(1 - \frac{1}{4} r_u^2)r_u \sin(\theta_u - \theta_v)$$

$$+ \chi_2 n_1^2 r_u \cos(\theta_u - \theta_v) - \chi_2 n_1^2 B \cos\theta_v\Big\}$$

(b) Autonomous System Derived When No External Resonance Occurs

When the difference between the frequency ω_0 of the self-excited oscillation and the driving frequency ω is large enough, the resulting oscillation may approximately be considered as a combination of two simple harmonic oscillations, one with ω_0, the other with ω. Then the solution of Eqs. (6.1) is written in the same form as Eqs. (4.16), i.e.,

$$\begin{aligned}
u(t) &= r_u(t) \cos [\omega_0 t + \theta_u(t)] + A_1 \cos \omega t \\
v(t) &= r_v(t) \cos [\omega_0 t + \theta_v(t)] + A_2 \cos \omega t \\
\dot{u}(t) &= - \omega_0 r_u(t) \sin [\omega_0 t + \theta_u(t)] - \omega A_1 \sin \omega t \\
\dot{v}(t) &= - \omega_0 r_v(t) \sin [\omega_0 t + \theta_v(t)] - \omega A_1 \sin \omega t
\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}
A_1 &= \frac{n_1^2(n_2^2 - \omega^2)}{(1 - \chi_1\chi_2)\omega^4 - (n_1^2 + n_2^2)\omega^2 + n_1^2 n_2^2} B \\
A_2 &= \frac{- \chi_2 n_1^2 \omega^2}{(1 - \chi_1\chi_2)\omega^4 - (n_1^2 + n_2^2)\omega^2 + n_1^2 n_2^2} B
\end{aligned} \tag{6.7}$$

Substituting Eqs. (6.6) into Eqs. (6.1) and using the averaging method as before, we obtain

$$\begin{aligned}
\dot{r}_u &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f_5(r_u, r_v, \theta_u, \theta_v, t) \sin (\omega_0 t + \theta_u) dt \\
\dot{r}_v &= \lim_{T\to\infty} \frac{1}{T} \int_0^T g_5(r_u, r_v, \theta_u, \theta_v, t) \sin (\omega_0 t + \theta_u) dt \\
r_u\dot{\theta}_u &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f_5(r_u, r_v, \theta_u, \theta_v, t) \cos (\omega_0 t + \theta_u) dt \\
r_v\dot{\theta}_v &= \lim \frac{1}{T} \int_0^T g_5(r_u, r_v, \theta_u, \theta_v, t) \cos (\omega_0 t + \theta_v) dt
\end{aligned} \tag{6.8}$$

where

$$f_5(r_u, r_v, \theta_u, \theta_v, t) = \frac{1}{\omega_0(1-\chi_1\chi_2)}\Big\{\mu\omega_0 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_u^2)r_u \sin(\omega_0 t + \theta_u)$$

$$+ [n_1^2 - (1-\chi_1\chi_2)\omega_0^2]r_u \cos(\omega_0 t + \theta_u)$$

$$- \mu\omega_0\chi_1 \frac{n_2^2}{n_1}\delta r_v \sin(\omega_0 t + \theta_v) + \chi_1 n_2^2 r_v \cos(\omega_0 t + \theta_v)$$

$$+ \mu\omega[n_1(1 - \tfrac{1}{4}A_1^2 - \tfrac{1}{2}r_u^2)A_1 - \chi_1 \frac{n_2^2}{n_1}\delta A_2] \sin\omega t$$

$$- \tfrac{1}{4}\mu\omega_0 n_1 r_u^3 \sin(3\omega_0 t + 3\theta_u) - \tfrac{1}{4}\mu\omega n_1 A_1^3 \sin 3\omega t$$

$$- \tfrac{1}{4}\mu(2\omega + \omega_0)n_1 A_1^2 r_u \sin[(2\omega + \omega_0)t + \theta_u]$$

$$- \tfrac{1}{4}\mu(2\omega - \omega_0)n_1 A_1^2 r_u \sin[(2\omega - \omega_0)t - \theta_u]$$

$$- \tfrac{1}{4}\mu(2\omega_0 + \omega)n_1 A_1 r_u^2 \sin[(2\omega_0 + \omega)t + 2\theta_u]$$

$$- \tfrac{1}{4}\mu(2\omega_0 - \omega)n_1 A_1 r_u^2 \sin[(2\omega_0 - \omega)t + 2\theta_u]\Big\} \tag{6.9}$$

$$g_5(r_u, r_v, \theta_u, \theta_v, t) = \frac{1}{\omega_0(1-\chi_1\chi_2)}\Big\{\mu\omega_0\chi_2 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_u^2)r_u \sin(\omega_0 t + \theta_u)$$

$$+ \chi_2 n_1^2 r_u \cos(\omega_0 t + \theta_u) - \mu\omega_0 \frac{n_2^2}{n_1}\delta r_v \sin(\omega_0 t + \theta_v)$$

$$+ [n_2^2 - (1-\chi_1\chi_2)\omega_0^2]r_v \cos(\omega_0 t + \theta_v)$$

$$+ \mu\omega[\chi_2 n_1(1 - \tfrac{1}{4}A_1^2 - \tfrac{1}{2}r_u^2)A_1 - \frac{n_2^2}{n_1}\delta A_2] \sin\omega t$$

$$- \tfrac{1}{4}\mu\omega_0\chi_2 n_1 r_u^3 \sin(3\omega_0 t + 3\theta_u) - \tfrac{1}{4}\mu\omega\chi_2 n_1 A_1^3 \sin 3\omega t$$

$$- \tfrac{1}{4}\mu(2\omega + \omega_0)\chi_2 n_1 A_1^2 r_u \sin[(2\omega + \omega_0)t + \theta_u]$$

$$- \tfrac{1}{4}\mu(2\omega - \omega_0)\chi_2 n_1 A_1^2 r_u \sin[(2\omega - \omega_0)t - \theta_u]$$

$$- \frac{1}{4}\mu(2\omega_0 + \omega)\chi_2 n_1 A_1 r_u^2 \sin[(2\omega_0 + \omega)t + 2\theta_u]$$

$$- \frac{1}{4}\mu(2\omega_0 - \omega)\chi_2 n_1 A_1 r_u^2 \sin[(2\omega_0 - \omega)t + 2\theta_u]\Big\}$$

Performing the integration of Eqs. (6.8) yields

$$\dot{r}_u = \frac{1}{2\omega_0(1 - \chi_1\chi_2)}\Big[\mu\omega_0 n_1(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_u^2)r_u + \chi_1 n_2^2 r_v \sin(\theta_u - \theta_v)$$

$$-\mu\omega_0\chi_1\frac{n_2^2}{n_1}\delta r_v \cos(\theta_u - \theta_v)\Big]$$

$$\dot{r}_v = \frac{1}{2\omega_0(1 - \chi_1\chi_2)}\Big[-\mu\omega_0\frac{n_2^2}{n_1}\delta r_v - \chi_2 n_1^2 r_u \sin(\theta_u - \theta_v)$$

$$+\mu\omega_0\chi_2 n_1(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_u^2)r_u \cos(\theta_u - \theta_v)\Big] \tag{6.10}$$

$$r_u\dot{\theta}_u = \frac{1}{2\omega_0(1 - \chi_1\chi_2)}\Big\{[n_1^2 - (1 - \chi_1\chi_2)\omega_0^2]r_u + \mu\omega_0\chi_1\frac{n_2^2}{n_1}\delta r_v \sin(\theta_u - \theta_v)$$

$$+ \chi_1 n_2^2 r_v \cos(\theta_u - \theta_v)\Big\}$$

$$r_v\dot{\theta}_v = \frac{1}{2\omega_0(1 - \chi_1\chi_2)}\Big\{[n_2^2 - (1 - \chi_1\chi_2)\omega_0^2]r_v$$

$$+\mu\omega_0\chi_2 n_1(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_u^2)r_u \sin(\theta_u - \theta_v) + \chi_2 n_1^2 r_u \cos(\theta_u - \theta_v)\Big\}$$

(c) Autonomous Systems Derived When $\omega = \omega_0/3$ or $\omega = 3\omega_0$

When the ratio between ω_0 and ω is in the neighborhood of an integer (different from unity) or a fraction, one may expect the higher-harmonic and subharmonic entrainments. As pointed out in Table 4.4 in Chap. 4, we consider the cases where ω is in the neighborhood of $\omega_0/3$ or $3\omega_0$. Then, the entrained oscillations of Eqs. (6.1) are written as

$$u(t) = r_u(t)\cos[n\omega t + \theta_u(t)] + A_1 \cos\omega t$$

$$v(t) = r_v(t)\cos[n\omega t + \theta_v(t)] + A_2\cos\omega t$$

$$\dot{u}(t) = -n\omega r_u(t)\sin[n\omega t + \theta_u(t)] - \omega A_1\sin\omega t \tag{6.11}$$

$$\dot{v}(t) = -n\omega r_v(t)\sin[n\omega t + \theta_v(t)] - \omega A_2\sin\omega t$$

where

$$n = 3 \quad \text{for} \quad \omega \cong \omega_0/3$$

$$n = 1/3 \quad \text{for} \quad \omega \cong 3\omega_0$$

The averaged equations are obtained from Eqs. (6.8) by replacing ω_0 by 3ω or $\omega/3$, respectively. Performing the integration of Eqs. (6.8) yields the following autonomous systems. When $\omega \cong \omega_0/3$, we obtain

$$\dot{r}_u = \frac{1}{6\omega(1-\chi_1\chi_2)}\Big[3\mu\omega n_1\big(1-\tfrac{1}{2}A_1^2-\tfrac{1}{4}r_u^2\big)r_u + \chi_1 n_2^2 r_v\sin(\theta_u-\theta_v)$$
$$- 3\mu\omega\chi_1\frac{n_2^2}{n_1}\delta r_v\cos(\theta_u-\theta_v) - \tfrac{1}{4}\mu\omega n_1 A_1^3\cos\theta_u\Big]$$

$$\dot{r}_v = \frac{1}{6\omega(1-\chi_1\chi_2)}\Big[-3\mu\omega\frac{n_2^2}{n_1}\delta r_v - \chi_2 n_1^2 r_u\sin(\theta_u-\theta_v)$$
$$+ 3\mu\omega\chi_2 n_1\big(1-\tfrac{1}{2}A_1^2-\tfrac{1}{4}r_u^2\big)r_u\cos(\theta_u-\theta_v) - \tfrac{1}{4}\mu\omega\chi_2 n_1 A_1^3\cos\theta_v\Big]$$

$$r_u\dot{\theta}_u = \frac{1}{6\omega(1-\chi_1\chi_2)}\Big\{[n_1^2 - 9(1-\chi_1\chi_2)\omega^2]r_u + 3\mu\omega\chi_1\frac{n_2^2}{n_1}\delta r_v\sin(\theta_u-\theta_v) \tag{6.12}$$
$$+ \chi_1 n_2^2 r_v\cos(\theta_u-\theta_v) + \tfrac{1}{4}\mu\omega n_1 A_1^3\sin\theta_u\Big\}$$

$$r_v\dot{\theta}_v = \frac{1}{6\omega(1-\chi_1\chi_2)}\Big\{[n_2^2 - 9(1-\chi_1\chi_2)\omega^2]r_v + 3\mu\omega\chi_2 n_1\big(1-\tfrac{1}{2}A_1^2-\tfrac{1}{4}r_u^2\big)r_u$$
$$\times\sin(\theta_u-\theta_v) + \chi_2 n_1^2 r_u\cos(\theta_u-\theta_v) + \tfrac{1}{4}\mu\omega\chi_2 n_1 A_1^3\sin\theta_v\Big\}$$

When $\omega \cong 3\omega_0$, we obtain

$$\dot{r}_u = \frac{3}{2\omega(1 - \chi_1\chi_2)} [\tfrac{1}{3}\mu\omega n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_u^2)r_u + \chi_1 n_2^2 r_v \sin(\theta_u - \theta_v)$$
$$- \tfrac{1}{3}\mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_v \cos(\theta_u - \theta_v) - \tfrac{1}{12}\mu\omega n_1 A_1 r_u^2 \cos 3\theta_u]$$
$$\dot{r}_v = \frac{3}{2\omega(1 - \chi_1\chi_2)} [-\tfrac{1}{3}\mu\omega \frac{n_2^2}{n_1}\delta r_v - \chi_2 n_1^2 r_u \sin(\theta_u - \theta_v)$$
$$+ \tfrac{1}{3}\mu\omega\chi_2 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_u^2)r_u \cos(\theta_u - \theta_v) - \tfrac{1}{12}\mu\omega\chi_2 n_1 A_1 r_u^2$$
$$\times \cos(2\theta_u + \theta_v)]$$
$$(6.13)$$
$$r_u\dot{\theta}_u = \frac{3}{2\omega(1 - \chi_1\chi_2)} \Big\{[n_1^2 - \tfrac{1}{9}(1 - \chi_1\chi_2)\omega^2]r_u + \tfrac{1}{3}\mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_v \sin(\theta_u - \theta_v)$$
$$+ \chi_1 n_2^2 r_v \cos(\theta_u - \theta_v) + \tfrac{1}{12}\mu\omega n_1 A_1 r_u^2 \sin 3\theta_u\Big\}$$
$$r_v\dot{\theta}_v = \frac{3}{2\omega(1 - \chi_1\chi_2)} \Big\{[n_2^2 - \tfrac{1}{9}(1 - \chi_1\chi_2)\omega^2]r_v + \tfrac{1}{3}\mu\omega\chi_2 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_u^2)$$
$$\times r_u \sin(\theta_u - \theta_v) + \chi_2 n_1^2 r_u \cos(\theta_u - \theta_v)$$
$$+ \tfrac{1}{12}\mu\omega\chi_2 n_1 A_1 r_u^2 \sin(2\theta_u + \theta_v)\Big\}$$

6.3 Harmonic Entrainment

(a) Steady-State Solutions

The steady states where r_u, r_v, θ_u, and θ_v are constant are obtained by equating $\dot{r}_u = \dot{r}_v = \dot{\theta}_u = \dot{\theta}_v = 0$ in Eqs. (6.5). Denoting the steady-state values of these variables by r_{u0}, r_{v0}, θ_{u0}, and θ_{v0}, respectively, we obtain

$$\mu\omega n_1(1 - \tfrac{1}{4}r_{u0}^2)r_{u0} - \chi_1 n_2^2 r_{v0} \sin(\theta_{u0} - \theta_{v0})$$
$$-\mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_{v0} \cos(\theta_{u0} - \theta_{v0}) - n_1^2 B \sin\theta_{u0} = 0$$

$$\mu\omega\frac{n_2^2}{n_1}\delta r_{v0} - \chi_2 n_1^2 r_{u0}\sin(\theta_{u0}-\theta_{v0})$$

$$+\mu\omega\chi_2 n_1(1-\tfrac{1}{4}r_{u0}^2)r_{u0}\cos(\theta_{u0}-\theta_{v0}) - \chi_2 n_1^2 B\sin\theta_{v0} = 0$$

$$[n_1^2-(1-\chi_1\chi_2)\omega^2]r_{u0} + \mu\omega\chi_1\frac{n_2^2}{n_1}\delta r_{v0}\sin(\theta_{u0}-\theta_{v0})$$

$$+\chi_1 n_2^2 r_{v0}\cos(\theta_{u0}-\theta_{v0}) - n_1^2 B\cos\theta_{u0} = 0 \tag{6.14}$$

$$[n_2^2-(1-\chi_1\chi_2)\omega^2]r_{v0} + \mu\omega\chi_2 n_1(1-\tfrac{1}{4}r_{u0}^2)r_{u0}\sin(\theta_{u0}-\theta_{v0})$$

$$+\chi_2 n_1^2 r_{u0}\cos(\theta_{u0}-\theta_{v0}) - \chi_2 n_1^2 B\cos\theta_{v0} = 0$$

Eliminating r_{v0}, θ_{u0}, and θ_{v0} from Eqs. (6.14) gives

$$\Big\{[\mu\omega n_1(\omega^2-n_2^2)(1-\tfrac{1}{4}r_{u0}^2) - \mu\omega\frac{n_2^2}{n_1}\delta(\omega^2-n_1^2)]^2 + [\mu^2\omega^2 n_2^2\delta(1-\tfrac{1}{4}r_{u0}^2)$$

$$+(\omega^2-n_1^2)(\omega^2-n_2^2)-\chi_1\chi_2\omega^4]^2\Big\}r_{u0}^2 - n_1^4B^2[\mu^2\omega^2\frac{n_2^4}{n_1^2}\delta^2+(\omega^2-n_2^2)^2]^2 = 0 \tag{6.15}$$

r_{v0}, θ_{u0}, and θ_{v0} are given by*

$$r_{v0}^2 = \frac{\chi_2^2\omega^4}{(\omega^2-n_2^2)^2+\mu^2\omega^2\frac{n_2^4}{n_1^2}\delta^2}\,r_{u0}^2$$

$$\sin\theta_{u0} = \frac{\mu\omega}{\chi_2 n_1^2 B r_{u0}}[\chi_2 n_1(1-\tfrac{1}{4}r_{u0}^2)r_{u0}^2 - \chi_1\frac{n_2^2}{n_1}\delta r_{v0}^2]$$

$$\cos\theta_{u0} = \frac{1}{\chi_2 n_1^2 B r_{u0}}[\chi_1(\omega^2-n_2^2)r_{v0}^2 - \chi_2(\omega^2-n_1^2)r_{u0}^2] \tag{6.16}$$

---

* The phase difference is given as follows

$$\cos(\theta_{u0}-\theta_{v0}) = \frac{(\omega^2-n_2^2)r_{v0}}{\omega^2\chi_2 r_{u0}}, \qquad \sin(\theta_{u0}-\theta_{v0}) = -\frac{\mu n_2^2\delta r_{v0}}{\omega\chi_2 n_1 r_{u0}}$$

$$\sin\theta_{v0} = \frac{\mu}{\omega\chi_2 n_1^2 B}\left[n_1(\omega^2 - n_2^2)(1 - \frac{1}{4} r_{u0}^2) - (\omega^2 - n_1^2)\frac{n_2^2}{n_1}\delta\right]r_{v0}$$

$$\cos\theta_{v0} = \frac{1}{\omega^2\chi_2 n_1^2 B}\left[\chi_1\chi_2\omega^4 - (\omega^2 - n_1^2)(\omega^2 - n_2^2) - \mu^2\omega^2 n_2^2\delta(1 - \frac{1}{4} r_{u0}^2)\right]r_{v0}$$

Putting B = 0 in Eqs. (6.15) yields

$$\begin{aligned} &\mu\omega n_1(\omega^2 - n_2^2)(1 - \frac{1}{4} r_{u0}^2) - \mu\omega\frac{n_2^2}{n_1}\delta(\omega^2 - n_1^2) = 0 \\ &\mu^2\omega^2 n_2^2\delta(1 - \frac{1}{4} r_{u0}^2) + (\omega^2 - n_1^2)(\omega^2 - n_2^2) - \chi_1\chi_2\omega^4 = 0 \end{aligned} \tag{6.17}$$

From Eqs. (6.17), we can derive the first member of Eqs. (3.28) and Eq. (3.29) which give the amplitude and the frequency of the self-excited oscillation.

(b) Stability Investigation

The stability of the solutions is tested by the behavior of small variations from the steady-state values, r_{u0}, r_{v0}, θ_{u0}, and θ_{v0}. In the same manner as in the preceding chapters, we obtain the characteristic equation (4.44), where

$$p = \frac{\mu n_1}{2(1 - \chi_1\chi_2)}\left[- 2 + 2(n_2/n_1)^2\delta + r_{u0}^2\right]$$

$$\begin{aligned} q = \frac{1}{4\omega^2(1 - \chi_1\chi_2)^2}\Big\{&\mu^2\omega^2 n_1^2\left[(1 - \frac{3}{4} r_{u0}^2)(1 - \frac{1}{4} r_{u0}^2) + (n_2/n_1)^4\delta^2\right. \\ &\left.- 2(n_2/n_1)^2\delta(2 - \chi_1\chi_2)(1 - \frac{1}{2} r_{u0}^2)\right] + \left[n_1^2 - (1 - \chi_1\chi_2)\omega^2\right]^2 \\ &+ \left[n_2^2 - (1 - \chi_1\chi_2)\omega^2\right]^2 + 2\chi_1\chi_2 n_1^2 n_2^2\Big\} \end{aligned}$$

$$\begin{aligned} r = \frac{\mu n_1}{4\omega^2(1 - \chi_1\chi_2)^2}\Big\{&\mu^2\omega^2 n_2^2\delta\left[(1 - \frac{3}{4} r_{u0}^2)(1 - \frac{1}{4} r_{u0}^2) - (n_2/n_1)^2\delta(1 - \frac{1}{2} r_{u0}^2)\right] \\ &+ (n_2/n_1)^2\delta\left[(n_1^2 - \omega^2)^2 + \omega^2\chi_1\chi_2(2n_1^2 - \omega^2)\right] \\ &- (1 - \frac{1}{2} r_{u0}^2)\left[(n_2^2 - \omega^2)^2 + \omega^2\chi_1\chi_2(2n_2^2 - \omega^2)\right]\Big\} \end{aligned} \tag{6.18}$$

$$s = \frac{1}{16\omega^4(1 - \chi_1\chi_2)^2}\Big\{\mu^2\omega^2 n_1^2[(n_2^2 - \omega^2)^2 + \mu^2\omega^2 n_1^2(n_2/n_1)^4\delta^4](1 - \tfrac{3}{4} r_{u0}^2)(1 - \tfrac{1}{4} r_{u0}^2)$$

$$+ \mu^2\omega^2 n_2^2\delta[2\omega^4\chi_1\chi_2(\tfrac{1}{2} r_{u0}^2 - 1) + (n_2/n_1)^2\delta(\omega^2 - n_1^2)^2]$$

$$+ [n_2^2(n_1^2 - \omega^2) - \omega^2(n_1^2 - \omega^2 + \chi_1\chi_2\omega^2)]^2\Big\}$$

The stability conditions are given by

$$p > 0, \qquad r > 0, \qquad s > 0, \qquad pqr - r^2 - p^2 s > 0 \tag{6.19}$$

(c) Numerical Examples

Equations (6.15) and (6.16) yield what we call the response curves for the harmonic oscillation as illustrated in Figs. 6.1 and 6.2. We consider the cases in which the system parameters are given by

$$\mu = 0.1 \qquad (n_2/n_1)^2\delta = 0.5 \qquad \text{and} \qquad n_2/n_1 = 1.0$$

and the coupling factor $k = \sqrt{\chi_1\chi_2}$ is 0.04 and 0.08. Figures 6.1a and 6.2a show the amplitude characteristics of the harmonic solution for different values of B. Similarly, Figs. 6.1b and 6.2b show the phase angles of the solution. The stability of the oscillations is tested by making use of conditions (6.19). The unstable portions of the response curves are shown dashed in Figs. 6.1 and 6.2. The fine lines in Figs. 6.1a and 6.2a are the stability limits given by*

$$s = 0, \qquad pqr - r^2 - p^2 s = 0 \tag{6.20}$$

It is readily verified that the vertical trangencies of the response curves occur

---

* As the results of numerical calculation, the curves obtained from these two relations are found to be effective among those derived from the conditions (6.19).

at the stability limit $s = 0$ (see Appendix II). In Figs. 6.1c and 6.2c are shown the regions of the harmonic entrainment on the $B\omega$ plane.

In Fig. 6. 1, where $k = 0.04$, the self-oscillatory system has only one self-excited oscillation of the frequency ω_0 ($= n_1$). In this case, the characteristics of Fig. 6.1 are similar to those of the system with one degree of freedom [10]. In Fig. 6.2, where $k = 0.08$, two different frequencies of the self-excited oscillations exist and the characteristics have two separate closed curves for small B.

6.4 Combination Oscillations without External Resonance

(a) Steady-State Solutions

When the difference between the driving frequency ω and the frequency ω_0 of the self-excited oscillation is large enough, the solution of the system is assumed as Eqs. (6.6). The steady-state solutions of the averaged equations (6.10) are obtained by equating $\dot{r}_u = \dot{r}_v = \dot{\theta}_u = \dot{\theta}_v = 0$, i.e.,

$$\begin{aligned}
&\mu\omega_0 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2)r_{u0} + \chi_1 n_2^2 r_{v0}\sin(\theta_{u0} - \theta_{v0}) \\
&\quad - \mu\omega_0\chi_1 \frac{n_2^2}{n_1}\delta r_{v0}\cos(\theta_{u0} - \theta_{v0}) = 0 \\
&-\mu\omega_0 \frac{n_2^2}{n_1}\delta r_{v0} - \chi_2 n_1^2 r_{u0}\sin(\theta_{u0} - \theta_{v0}) \\
&\quad + \mu\omega_0\chi_2 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2)r_{u0}\cos(\theta_{u0} - \theta_{v0}) = 0 \\
&[n_1^2 - (1 - \chi_1\chi_2)\omega_0^2]r_{u0} + \mu\omega_0\chi_1\frac{n_2^2}{n_1}\delta r_{v0}\sin(\theta_{u0} - \theta_{v0}) \\
&\quad + \chi_1 n_2^2 r_{v0}\cos(\theta_{u0} - \theta_{v0}) = 0 \\
&[n_2^2 - (1 - \chi_1\chi_2)\omega_0^2]r_{v0} + \mu\omega_0\chi_2 n_1(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2)r_{u0}\sin(\theta_{u0} - \theta_{v0}) \\
&\quad + \chi_2 n_1^2 r_{u0}\cos(\theta_{u0} - \theta_{v0}) = 0
\end{aligned} \tag{6.21}$$

We see, from Eqs. (6.21), that there are two types of the steady states, i.e.,

$$\begin{aligned} &(1) \quad r_{u0} = r_{v0} = 0 \\ &(2) \quad r_{u0} \neq 0, \quad r_{v0} \neq 0 \end{aligned} \tag{6.22}$$

The steady state (1) corresponds to the periodic oscillation of frequency ω, i.e., the harmonic entrainment. In the steady state (2), the solution has the two frequency components ω and ω_0. Therefore, the combination oscillation occurs. In this case, after some algebraic manipulation, we obtain the amplitudes r_{u0}, r_{v0}, and the phase difference $\theta_{u0} - \theta_{v0}$, i.e.,

$$r_{u0}^2 = (\rho_u - 2A_1^2)$$

where (6.23)

$$\rho_u = 4\left(1 - \frac{\omega_0^2 - n_1^2}{\omega_0^2 - n_2^2}\frac{n_2^2}{n_1^2}\delta\right)$$

and

$$r_{v0}^2 = \frac{\chi_2(\omega_0^2 - n_1^2)}{\chi_1(\omega_0^2 - n_2^2)} r_{u0}^2 = \frac{\chi_2 n_1^2}{\chi_1 n_2^2 \delta} r_{u0}^2\left(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2\right)$$

$$\sin(\theta_{u0} - \theta_{v0}) = -\frac{\mu}{\omega_0\chi_2}\frac{n_2^2}{n_1}\delta\frac{r_{v0}}{r_{u0}} = -\frac{\mu n_1}{\omega_0\chi_1}\frac{r_{u0}}{r_{v0}}\left(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2\right) \tag{6.24}$$

$$\cos(\theta_{u0} - \theta_{v0}) = \frac{\omega_0^2 - n_2^2}{\omega_0^2\chi_2}\frac{r_{v0}}{r_{u0}} = \frac{\omega_0^2 - n_1^2}{\omega_0^2\chi_1}\frac{r_{u0}}{r_{v0}}$$

The frequency ω_0 is the root of the following equation which is obtained by eliminating r_{u0}, r_{v0}, and $\theta_{u0} - \theta_{v0}$ from Eqs. (6.21),

$$[(1 - \chi_1\chi_2)\omega_0^4 - (n_1^2 + n_2^2)\omega_0^2 + n_1^2 n_2^2](\omega_0^2 - n_2^2) + \left(\mu\frac{n_2^2}{n_1}\delta\right)^2(\omega_0^2 - n_1^2)\omega_0^2 = 0 \tag{6.25}$$

Equation (6.25) is identical with Eq. (3.29) in Chap. 3.

(b) Stability Investigation

The stability of the steady-state solution (2) of Eqs. (6.22) is tested in much the same way as before. The coefficients of the variational equations are given by

$$
\begin{aligned}
a_{11} &= K\mu\omega_0 n_1\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{3}{4}r_{u0}^2\right) \\
a_{12} &= -K\frac{\mu\omega_0\chi_1}{\chi_2}\frac{n_2^2}{n_1}\delta\frac{r_{v0}}{r_{u0}} \\
a_{13} &= -a_{14} = -K\frac{\chi_1}{\omega_0^2\chi_2}\left[\left(\mu\omega_0\frac{n_2^2}{n_1}\delta\right)^2 - n_2^2(\omega_0^2 - n_2^2)\right]\frac{r_{v0}^2}{r_{u0}} \\
a_{21} &= K\frac{\mu n_1^2}{\omega_0}\left[\frac{n_2^2}{n_1}\delta + (\omega_0^2 - n_2^2)\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{3}{4}r_{u0}^2\right)\right]\frac{r_{v0}}{r_{u0}} \\
a_{22} &= -K\mu\omega_0\frac{n_2^2}{n_1}\delta \\
a_{23} &= -a_{24} = \frac{K}{\omega_0^2}\left[-n_1^2(\omega_0^2 - n_1^2) + \mu^2\omega_0^2 n_2^2\delta\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2\right)\right]r_{v0} \\
a_{31} &= K\frac{\chi_1}{\omega_0^2\chi_2}\left[\left(\mu\omega_0\frac{n_2^2}{n_1}\delta\right)^2 - n_2^2(\omega_0^2 - n_2^2)\right]\frac{r_{v0}^2}{r_{u0}^3} \qquad (6.26)\\
a_{32} &= -\frac{r_{u0}}{r_{v0}}a_{31} \\
a_{33} &= -a_{34} = K\frac{\mu\omega_0\chi_1}{\chi_2}\frac{n_2^2}{n_1}\delta\frac{r_{v0}^2}{r_{u0}^2} \\
a_{41} &= \frac{K}{\omega_0^2}\left[-\mu^2\omega_0^2 n_2^2\delta\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{3}{4}r_{u0}^2\right) + n_1^2(\omega_0^2 - n_2^2)\right]\frac{1}{r_{u0}} \\
a_{42} &= \frac{K}{\omega_0^2}\left[\mu^2\omega_0^2 n_2^2\delta\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2\right) - n_1^2(\omega_0^2 - n_2^2)\right]r_{v0} \\
a_{43} &= -a_{44} = K\frac{\mu n_1}{\omega_0}\left[(\omega_0^2 - n_2^2)\left(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2\right) + n_2^2\delta\right]
\end{aligned}
$$

where

$$K = \frac{1}{2\omega_0(1 - \chi_1\chi_2)}$$

Since $a_{i4} = - a_{i3}$ $(i = 1, .., 4)$, the characteristic equation is reduced to

$$\lambda(\lambda^3 + p\lambda^2 + q\lambda + r) = 0 \tag{6.27}$$

where p, q, and r are given by Eqs. (3.19). The stability conditions are as follows:

$$p > 0, \qquad q > 0, \qquad pq - r > 0 \tag{6.28}$$

The stability conditions for the steady state (1) of Eqs. (6.22) cannot be obtained by using Eqs. (6.26), since they contain r_{u0}, r_{v0} in the denominater. Let ξ_u and ξ_v be the small amplitudes and θ_u and θ_v be the phase angles of the small variations from the solution u = v = 0 [cf. Eqs. (4.87)]. Replacing r_u and r_v in Eqs. (6.10) by ξ_u and ξ_v, respectively, and discarding the terms of order higher than the first in ξ_u and ξ_v, we obtain the following equations.

$$\begin{aligned}
\dot{\xi}_u &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} [\mu\omega_0 n_1(1 - \tfrac{1}{2} A_1^2)\xi_u + \chi_1 n_2^2 \xi_v \sin(\theta_u - \theta_v) \\
&\qquad - \mu\omega_0\chi_1 \frac{n_2^2}{n_1} \delta\xi_v \cos(\theta_u - \theta_v)] \\
\dot{\xi}_v &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} [-\mu\omega_0 \frac{n_2^2}{n_1}\delta\xi_v - \chi_2 n_1^2 \xi_u \sin(\theta_u - \theta_v) \\
&\qquad + \mu\omega_0\chi_2 n_1(1 - \tfrac{1}{2} A_1^2)\xi_u \cos(\theta_u - \theta_v)] \\
\xi_u\dot{\theta}_u &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} \Big\{[n_1^2 - (1 - \chi_1\chi_2)\omega_0^2]\xi_u + \mu\omega_0\chi_1 \frac{n_2^2}{n_1}\delta\xi_v \sin(\theta_u - \theta_v) \\
&\qquad + \chi_1 n_2^2 \xi_v \cos(\theta_u - \theta_v)\Big\} \\
\xi_v\dot{\theta}_v &= \frac{1}{2\omega_0(1 - \chi_1\chi_2)} \Big\{[n_2^2 - (1 - \chi_1\chi_2)\omega_0^2]\xi_v + \mu\omega_0\chi_2 n_1(1 - \tfrac{1}{2} A_1^2)\xi_u \sin(\theta_u - \theta_v) \\
&\qquad + \chi_2 n_1^2 \xi_u \cos(\theta_u - \theta_v)\Big\}
\end{aligned} \tag{6.29}$$

We introduce new variables a_1, a_2, and b_1, b_2 defined by

$$\begin{aligned} a_1 &= \xi_u \cos\theta_u, \qquad a_2 = \xi_v \cos\theta_v \\ b_1 &= -\xi_u \sin\theta_u, \quad b_2 = -\xi_v \sin\theta_v \end{aligned} \tag{6.30}$$

Substituting Eqs. (6.30) into Eqs. (6.29) yields the variational equations with respect to a_i and b_i ($i = 1, 2$). Then, the characteristic equation is given by

$$\begin{vmatrix} \frac{\mu n_1}{2(1-\chi_1\chi_2)}(1-\frac{1}{2}A_1^2)-\lambda & \frac{n_1^2-(1-\chi_1\chi_2)\omega_0^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{-\mu\chi_1 n_2^2\delta}{2(1-\chi_1\chi_2)n_1} & \frac{\chi_1 n_2^2}{2\omega_0(1-\chi_1\chi_2)} \\ -\frac{n_1^2-(1-\chi_1\chi_2)\omega_0^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{\mu n_1}{2(1-\chi_1\chi_2)}(1-\frac{1}{2}A_1^2)-\lambda & \frac{-\chi_1 n_2^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{-\mu\chi_1 n_2^2\delta}{2(1-\chi_1\chi_2)n_1} \\ \frac{\mu\chi_2 n_1}{2(1-\chi_1\chi_2)}(1-\frac{1}{2}A_1^2) & \frac{\chi_2 n_1^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{-\mu n_2^2\delta}{2(1-\chi_1\chi_2)n_1}-\lambda & \frac{n_2^2-(1-\chi_1\chi_2)\omega_0^2}{2\omega_0(1-\chi_1\chi_2)} \\ \frac{-\chi_2 n_1^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{\mu\chi_2 n_1}{2(1-\chi_1\chi_2)}(1-\frac{1}{2}A_1^2) & -\frac{n_2^2-(1-\chi_1\chi_2)\omega_0^2}{2\omega_0(1-\chi_1\chi_2)} & \frac{-\mu n_2^2\delta}{2(1-\chi_1\chi_2)n_1}-\lambda \end{vmatrix}$$

$$= [\lambda^2 + (p_1 + ip_2)\lambda + (q_1 + iq_2)][\lambda^2 + (p_1 - ip_2)\lambda + (q_1 - iq_2)]$$

$$= 0 \tag{6.31}$$

where

$$\begin{aligned} i &= \sqrt{-1} \\ p_1 &= -\frac{\mu n_1}{2(1-\chi_1\chi_2)}\left(1 - \frac{1}{2}A_1^2 - \frac{n_2^2}{n_1^2}\delta\right) \\ p_2 &= -\frac{1}{2\omega_0(1-\chi_1\chi_2)}[n_1^2 + n_2^2 - 2(1-\chi_1\chi_2)\omega_0^2] \\ q_1 &= -\frac{\mu^2 n_2^2\delta}{16(1-\chi_1\chi_2)}(\rho_u - 2A_1^2) \\ q_2 &= \frac{\mu n_1(n_2^2-\omega_0^2)}{16\omega_0(1-\chi_1\chi_2)^2}(\rho_u - 2A_1^2) \end{aligned} \tag{6.32}$$

The conditions for stability are that the real parts of characteristic roots λ's of Eqs. (6.31) are all negative. After some algebraic manipulation, these conditions become*

$$p_1 > 0, \qquad p_1^2 q_1 + p_1 p_2 q_2 - q_2^2 > 0 \tag{6.33}$$

Substituting Eqs. (6.32) into (6.33) yields

$$1 - \frac{1}{2} A_1^2 - \left(\frac{n_2}{n_1}\right)^2 < 0$$

and (6.34)

$$[1 - \frac{1}{2} A_1^2 - \left(\frac{n_2}{n_1}\right)^2 \delta]^2 [\chi_1 \chi_2 n_1^2 n_2^2 - \mu^2 \omega_0^2 n_2^2 \delta (1 - \chi_1 \chi_2)(1 - \frac{1}{2} A_1^2)]$$

$$+ [(1 - \frac{1}{2} A_1^2)(n_2^2 - n_1^2 - \chi_1 \chi_2 \omega_0^2) + \chi_1 \chi_2 \omega_0^2 \left(\frac{n_2}{n_1}\right)^2 \delta]$$

$$\times [\chi_1 \chi_2 \omega_0^2 (1 - \frac{1}{2} A_1^2) + (n_1^2 - n_2^2 - \chi_1 \chi_2 \omega_0^2)\left(\frac{n_2}{n_1}\right)^2 \delta] > 0$$

6.5 Higher-Harmonic Entrainment

(a) Steady-State Solutions

The steady-state solutions of Eqs. (6.12) are obtained by equating $\dot{r}_u = \dot{r}_v = \dot{\theta}_u = \dot{\theta}_v = 0$.

$$3\mu\omega n_1 (1 - \frac{1}{2} A_1^2 - \frac{1}{4} r_{u0}^2) r_{u0} + \chi_1 n_2^2 r_{v0} \sin(\theta_{u0} - \theta_{v0})$$
$$- 3\mu\omega\chi_1 \frac{n_2^2}{n_1} \delta r_{v0} \cos(\theta_{u0} - \theta_{v0}) - \frac{1}{4}\mu\omega n_1 A_1^3 \cos\theta_{u0} = 0$$

$$- 3\mu\omega \frac{n_2^2}{n_1} \delta r_{v0} - \chi_2 n_1^2 r_{u0} \sin(\theta_{u0} - \theta_{v0})$$
$$+ 3\mu\omega\chi_2 n_1 (1 - \frac{1}{2} A_1^2 - \frac{1}{4} r_{u0}^2) r_{u0} \cos(\theta_{u0} - \theta_{v0}) - \frac{1}{4}\mu\omega\chi_2 n_1 A_1^3 \cos\theta_{v0} = 0$$

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* See the footnote of p. 74.

$$[n_1^2 - 9(1 - \chi_1\chi_2)\omega^2]r_{u0} + 3\mu\omega\chi_1 \frac{n_2^2}{n_1}\delta r_{v0} \sin(\theta_{u0} - \theta_{v0}) \tag{6.35}$$

$$+ \chi_1 n_2^2 r_{v0} \cos(\theta_{u0} - \theta_{v0}) + \frac{1}{4}\mu\omega n_1 A_1^3 \sin\theta_{u0} = 0$$

$$[n_2^2 - 9(1 - \chi_1\chi_2)\omega^2]r_{v0} + 3\mu\omega\chi_2 n_1(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)r_{u0} \sin(\theta_{u0} - \theta_{v0})$$

$$+ \chi_2 n_1^2 r_{u0} \cos(\theta_{u0} - \theta_{v0}) + \frac{1}{4}\mu\omega\chi_2 n_1 A_1^3 \sin\theta_{v0} = 0$$

Eliminating r_{v0}, θ_{u0}, and θ_{v0} from Eqs. (6.35), we obtain r_{u0} by

$$\Big\{[9\mu^2\omega^2 n_1^2(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)^2 + (n_1^2 - 9\omega^2)^2][9\mu^2\omega^2\Big(\frac{n_2^2}{n_1}\Big)^2\delta^2 + (n_2^2 - 9\omega^2)^2]$$

$$- 162\omega^4\chi_1\chi_2[(n_1^2 - 9\omega^2)(n_2^2 - 9\omega^2) + 9\mu^2\omega^2 n_2^2\delta(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)]$$

$$+ (81\omega^4\chi_1\chi_2)^2\Big\}r_{u0}^2 - \frac{1}{16}\mu^2\omega^2 n_1^2 A_1^6[9\mu^2\omega^2\Big(\frac{n_2^2}{n_1}\Big)^2\delta^2 + (n_2^2 - 9\omega^2)^2] = 0 \tag{6.36}$$

r_{v0}, θ_{u0}, and θ_{v0} are given by*

$$r_{v0}^2 = \frac{81\omega^4\chi_2^2}{9\mu^2\omega^2\Big(\frac{n_2^2}{n_1}\Big)^2\delta^2 + (n_2^2 - 9\omega^2)^2} r_{u0}^2$$

$$\sin\theta_{u0} = \frac{4}{\mu\omega\chi_2 n_1 A_1^3 r_{u0}}[-\chi_2(n_1^2 - 9\omega^2)r_{u0}^2 + \chi_1(n_2^2 - 9\omega^2)r_{v0}^2]$$

$$\cos\theta_{u0} = \frac{12}{\chi_2 A_1^3 r_{u0}}[\chi_2(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)r_{u0}^2 - \chi_1\Big(\frac{n_2}{n_1}\Big)^2\delta r_{v0}^2] \tag{6.37}$$

$$\sin\theta_{v0} = \frac{4}{9\mu\omega^3\chi_2 n_1 A_1^3}[(n_1^2 - 9\omega^2)(n_2^2 - 9\omega^2)$$

$$+ 9\mu^2\omega^2 n_2^2\delta(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2) - 81\omega^4\chi_1\chi_2]r_{v0}$$

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* The phase difference is given by

$$\sin(\theta_{u0} - \theta_{v0}) = -\frac{\mu r_{v0}}{3\omega\chi_2 r_{u0}}\frac{n_2^2}{n_1}\delta, \qquad \cos(\theta_{u0} - \theta_{v0}) = -\frac{r_{v0}}{9\omega^2\chi_2 r_{u0}}(n_2^2 - 9\omega^2)$$

$$\cos\theta_{v0} = \frac{4}{3\omega^2\chi_2 A_1^3}\left[\left(\frac{n_2}{n_1}\right)^2\delta(n_1^2 - 9\omega^2) - (1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)(n_2^2 - 9\omega^2)\right]r_{v0}$$

We see, from Eqs. (6.11), that the solution is periodic with components of frequencies ω and 3ω.

(b) Stability Investigation

The stability conditions are the same as (6.19). The coefficients a_{ij} of the variational equations (4.42) are given by

$$\begin{aligned}
a_{11} &= 3K\mu\omega n_1(1 - \frac{1}{2}A_1^2 - \frac{3}{4}r_{u0}^2) \\
a_{12} &= 3K\mu\omega\frac{\chi_2}{\chi_1}\frac{n_2^2}{n_1}\delta\frac{r_{v0}}{r_{u0}} \\
a_{13} &= -Kr_{u0}[n_1^2 - 9(1 - \chi_1\chi_2)\omega^2] \\
a_{14} &= K\frac{\chi_1}{\chi_2}\frac{r_{v0}^2}{r_{u0}}\left[\left(\frac{n_2}{3\omega}\right)^2(n_2^2 - 9\omega^2) + (\mu\frac{n_2^2}{n_1}\delta)^2\right] \\
a_{21} &= K\frac{\mu n_1}{3\omega}[n_2^2\delta - (n_2^2 - 9\omega^2)(1 - \frac{1}{2}A_1^2 - \frac{3}{4}r_{u0}^2)]\frac{r_{v0}}{r_{u0}} \\
a_{22} &= -3K\mu\omega\frac{n_2^2}{n_1}\delta \\
a_{23} &= Kn_1^2\left[\frac{n_2^2 - 9\omega^2}{9\omega^2} + \mu^2\left(\frac{n_2}{n_1}\right)^2\delta(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2)\right]r_{v0} \\
a_{24} &= -K[n_2^2 - 9(1 - \chi_1\chi_2)\omega^2]r_{v0} \\
a_{31} &= K[n_1^2 - 9(1 - \chi_1\chi_2)\omega^2]\frac{1}{r_{u0}} \\
a_{32} &= -K\frac{\chi_1}{\chi_2}\left[(\mu\frac{n_2^2}{n_1}\delta)^2 + \left(\frac{n_2}{3\omega}\right)^2(n_2^2 - 9\omega^2)\right]\frac{r_{v0}}{r_{u0}^2} \\
a_{33} &= 3K\mu\omega n_1(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2) \\
a_{34} &= -K\mu\omega n_1\frac{\chi_1}{\chi_2}\frac{n_2^2}{n_1}\delta\frac{r_{v0}^2}{r_{u0}^2}
\end{aligned} \tag{6.38}$$

$$a_{41} = -K[\mu^2 n_2^2\delta(1 - \tfrac{1}{2}A_1^2 - \tfrac{3}{4}r_{u0}^2) + \left(\frac{n_1}{3\omega}\right)^2(n_2^2 - 9\omega^2)]\frac{1}{r_{u0}}$$

$$a_{42} = K[n_2^2 - 9(1 - \chi_1\chi_2)\omega^2]\frac{1}{r_{v0}}$$

$$a_{43} = K\frac{\mu n_1}{3\omega}[(n_2^2 - 9\omega^2)(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2) + n_2^2\delta]$$

$$a_{44} = -3K\mu\omega\frac{n_2^2}{n_1}\delta$$

where

$$K = \frac{1}{6\omega(1 - \chi_1\chi_2)} .$$

(c) Numerical Examples

Numerical analyses of the response characteristics for the higher-harmonic oscillation were carried out for the same values of the system parameters as those in Sec. 6.3c, i.e.,

$$\mu = 0.1 \qquad (n_2/n_1)^2\delta = 0.5 \qquad n_2/n_1 = 1.0 \qquad \text{and} \qquad k = 0.04,\ 0.08$$

With these values of the parameters, the response characteristics of the amplitudes r_{u0}, r_{v0}, and the phase angles θ_{u0}, θ_{v0} are calculated by using Eqs. (6.36) and (6.37), and illustrated in Figs. a and b of 6.3 and 6.4. The stability limits are obtained from (6.20) by using Eqs. (6.38). The vertical tangencies of the response curves occur at the stability limit $s = 0$. The dashed portions of the response curves show the unstable oscillations. The regions of the higher-harmonic entrainment on the $B\omega$ plane are also shown in Figs. 6.3c and 6.4c. Hence, when the amplitude B and the frequency ω of the external force are given at any point inside these regions, the self-excited oscillation is entrained by the frequency which is 3 times the driving frequency.

6.6 Subharmonic Entrainment

(a) Steady-State Solutions

The steady-state solutions of Eqs. (6.13) are obtained by equating $\dot{r}_u = \dot{r}_v = \dot{\theta}_u = \dot{\theta}_v = 0$. We can easily see that there are two types of steady-state solutions, i.e.,

$$\begin{aligned} &(1) \quad r_{u0} = 0, \qquad r_{v0} = 0 \\ &(2) \quad r_{u0} \neq 0, \qquad r_{v0} \neq 0 \end{aligned} \tag{6.39}$$

The solution is periodic of frequency ω in (1). Therefore the harmonic entrainment occurs. In the steady state (2), the solution is periodic of frequencies ω and $\omega/3$. Therefore the subharmonic entrainment occurs.

The harmonic entrainment has already been discussed in Sec. 6.4. In the steady state (2), after some algebraic manipulation, we obtain r_{u0} by

$$\begin{aligned} &[\tfrac{1}{9}\mu^2\omega^2 n_1^2(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2)^2 + (n_1^2 - \tfrac{\omega^2}{9})^2][\tfrac{1}{9}\mu^2\omega^2\left(\frac{n_2^2}{n_1}\right)^2\delta^2 + (n_2^2 - \tfrac{\omega^2}{9})^2] \\ &- \tfrac{2}{81}\omega^4\chi_1\chi_2[9\mu^2\omega^2 n_2^2\delta(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2) + (n_1^2 - \tfrac{\omega^2}{9})(n_2^2 - \tfrac{\omega^2}{9})] \\ &+ (\tfrac{\omega^4}{81}\chi_1\chi_2)^2 = \tfrac{1}{144}\mu^2\omega^2 n_1^2 A_1^2 r_{u0}^2[\tfrac{1}{9}\mu^2\omega^2\left(\frac{n_2^2}{n_1}\right)^2\delta^2 + (n_2^2 - \tfrac{\omega^2}{9})^2] \end{aligned} \tag{6.40}$$

Then, r_{v0}, θ_{u0}, and θ_{v0} are given by

$$\begin{aligned} r_{v0}^2 &= \frac{\omega^4\chi_2^2}{81[\frac{1}{9}\mu^2\omega^2\left(\frac{n_2^2}{n_1}\right)^2\delta^2 + (n_2^2 - \frac{\omega^2}{9})^2]} r_{u0}^2 \\ \sin 3\theta_{u0} &= \frac{12}{\mu\omega\chi_2 n_1 A_1 r_{u0}^3}[-\chi_2(n_1^2 - \tfrac{\omega^2}{9})r_{u0}^2 + \chi_1(n_2^2 - \tfrac{\omega^2}{9})r_{v0}^2] \\ \cos 3\theta_{u0} &= \frac{4}{\chi_2 A_1 r_{u0}^3}[\chi_2(1 - \tfrac{1}{2}A_1^2 - \tfrac{1}{4}r_{u0}^2)r_{u0}^2 - \chi_1\left(\frac{n_2}{n_1}\right)^2\delta r_{v0}^2] \end{aligned} \tag{6.41}$$

$$\sin(\theta_{u0} - \theta_{v0}) = -\frac{3\mu}{\omega\chi_2}\frac{r_{v0}}{r_{u0}}\frac{n_2^2}{n_1}\delta$$

$$\cos(\theta_{u0} - \theta_{v0}) = -\frac{9}{\omega^2\chi_2}\frac{r_{v0}}{r_{u0}}(n_2^2 - \frac{\omega^2}{9})$$

(b) Stability Investigation

The stability conditions for the steady state (1) of Eqs. (6.39) are the same as the conditions (6.34) which are obtained in Sec. 6.4b.

The stability conditions for the steady state (2) of Eqs. (6.39) are given by (6.19). The variational equations (4.42) are sought from Eqs. (6.13). Their coefficients a_{ij} are

$$a_{11} = \frac{K}{3}\mu\omega n_1[2\frac{\chi_2}{\chi_1}\left(\frac{n_2}{n_1}\right)^2\delta\left(\frac{r_{v0}}{r_{u0}}\right)^2 - (1 - \frac{1}{2}A_1^2 + \frac{1}{4}r_{u0}^2)]$$

$$a_{12} = -\frac{K}{3}\mu\omega\frac{n_2^2}{n_1}\delta\frac{r_{v0}}{r_{u0}}$$

$$a_{13} = -K[\mu^2\frac{\chi_1}{\chi_2}\frac{r_{v0}^2}{r_{u0}}\left(\frac{n_2^2}{n_1}\delta\right)^2 + \frac{9}{\omega^2}\frac{\chi_1}{\chi_2}\frac{r_{v0}^2}{r_{u0}}(n_2^2 - \frac{\omega^2}{9})(n_2^2 - \frac{\omega^2}{3}) + 3r_{u0}(n_1^2 - \frac{\omega^2}{9})]$$

$$a_{14} = K\frac{\chi_1}{\chi_2}\frac{r_{v0}^2}{r_{u0}}[\left(\frac{3n_2}{\omega}\right)^2(n_2^2 - \frac{\omega^2}{9}) + (\mu\frac{n_2^2}{n_1}\delta)^2]$$

$$a_{21} = 3K\frac{\mu}{\omega}\frac{r_{v0}}{r_{u0}}[-\left(\frac{n_2}{n_1}\right)^2\delta(n_1^2 - \frac{2}{9}\omega^2) + n_1(n_2^2 - \frac{\omega^2}{9})(1 - \frac{1}{2}A_1^2 + \frac{1}{4}r_{u0}^2)]$$

$$a_{22} = -\frac{K}{3}\mu\omega\frac{n_2^2}{n_1}\delta$$

$$a_{23} = Kr_{v0}[\frac{9}{\omega^2}(n_2^2 - \frac{\omega^2}{9})(3n_1^2 - \frac{2}{9}\omega^2) + 3\mu^2 n_2^2\delta(1 - \frac{1}{2}A_1^2 - \frac{1}{4}r_{u0}^2) - \frac{2}{9}\chi_1\chi_2\omega^2]$$

$$a_{24} = -Kr_{v0}[n_2^2 - \frac{\omega^2}{9}(1 - \chi_1\chi_2)]$$

$$a_{31} = K \frac{1}{r_{u0}} [\mu^2 \frac{\chi_1}{\chi_2} \left(\frac{r_{v0}}{r_{u0}}\right)^2 (\frac{n_2^2}{n_1} \delta)^2 + \frac{\chi_1}{\chi_2} \left(\frac{r_{v0}}{r_{u0}}\right)^2 \frac{9}{\omega^2} (n_2^2 - \frac{\omega^2}{9})(n_2^2 + \frac{\omega^2}{9}) \tag{6.42}$$

$$- (n_1^2 - \frac{\omega^2}{9})]$$

$$a_{32} = - K \frac{\chi_1}{\chi_2} \frac{r_{v0}}{r_{u0}^2} [(\mu \frac{n_2^2}{n_1} \delta)^2 + \left(\frac{3n_2}{\omega}\right)^2 (n_2^2 - \frac{\omega^2}{9})]$$

$$a_{33} = K\mu\omega[- \frac{2}{3} \frac{\chi_1}{\chi_2} \left(\frac{r_{v0}}{r_{u0}}\right)^2 \frac{n_2^2}{n_1} \delta + n_1(1 - \frac{1}{2} A_1^2 - \frac{1}{4} r_{u0}^2)]$$

$$a_{34} = \frac{K}{3} \mu\omega \frac{\chi_2}{\chi_1} \left(\frac{r_{v0}}{r_{u0}}\right)^2 \frac{n_2^2}{n_1} \delta$$

$$a_{41} = K[\mu^2 n_2^2 \delta \frac{1}{r_{u0}} (1 - \frac{1}{2} A_1^2 + \frac{1}{4} r_{u0}^2) + \frac{9}{\omega^2} \frac{1}{r_{u0}} (n_2^2 - \frac{\omega^2}{9})(n_1^2 - \frac{2\omega^2}{9})$$

$$- \frac{2}{9} \frac{\omega^2}{r_{u0}} \chi_1 \chi_2]$$

$$a_{42} = K \frac{1}{r_{v0}} [n_2^2 - \frac{\omega^2}{9} (1 - \chi_1 \chi_2)]$$

$$a_{43} = 3K \frac{\mu n_1}{\omega} [- 3(n_2^2 - \frac{\omega^2}{9})(1 - \frac{1}{2} A_1^2 - \frac{1}{4} r_{u0}^2) + \left(\frac{n_2}{n_1}\right)^2 \delta (3n_1^2 - \frac{2}{9} \omega^2)]$$

$$a_{44} = - \frac{K}{3} \mu\omega \frac{n_2^2}{n_1} \delta$$

where

$$K = \frac{3}{2\omega(1 - \chi_1 \chi_2)}$$

(c) Numerical Examples

Numerical analyses of the response characteristics for the subharmonic oscillation were carried out by using the same parameters as in Secs. 6.3c and 6.5c, i.e.,

$$\mu = 0.1 \qquad \left(\frac{n_2}{n_1}\right)^2 \delta = 0.5 \qquad \frac{n_2}{n_1} = 1.0 \qquad \text{and} \qquad k = 0.04, 0.08$$

The response characteristics are calculated by using Eqs. (6.40) and (6.41), and illustrated in Figs. 6.5 and 6.6. The stability of oscillation is tested by using (6.19) and Eqs. (6.42). The unstable responses are shown dashed in these figures. The regions of the subharmonic entrainment on the $B\omega$ plane are shown in Figs. 6.5c and 6.6c. In Fig. 6.5c, where $k = 0.04$, the region of entrainment is closed one. On the other hand, with increase of value of k, this region separates into two closed regions as seen in Figs. 6.6c. When B and ω are given in these regions the subharmonic oscillation occurs.

6.7 Concluding Remarks

We have investigated the forced oscillations in a self-oscillatory system whose natural frequencies are close each other. The phenomenon of frequency entrainment occurs at the harmonic (i.e., fundamental), higher-harmonic, or subharmonic frequency of the external force. The amplitude and phase characteristics of the entrained oscillations are obtained, and the regions of entrainment are shown. When the coupling between two resonant circuits is small, the regions of harmonic, higher-harmonic, and subharmonic entrainments are similar to those in a system with one degree of freedom. As the coupling becomes larger, these regions are devided into two parts.

The almost periodic oscillation which develops from the entrained oscillation is correlated with a limit cycle in the phase space. This type of oscillation is not treated in this study.

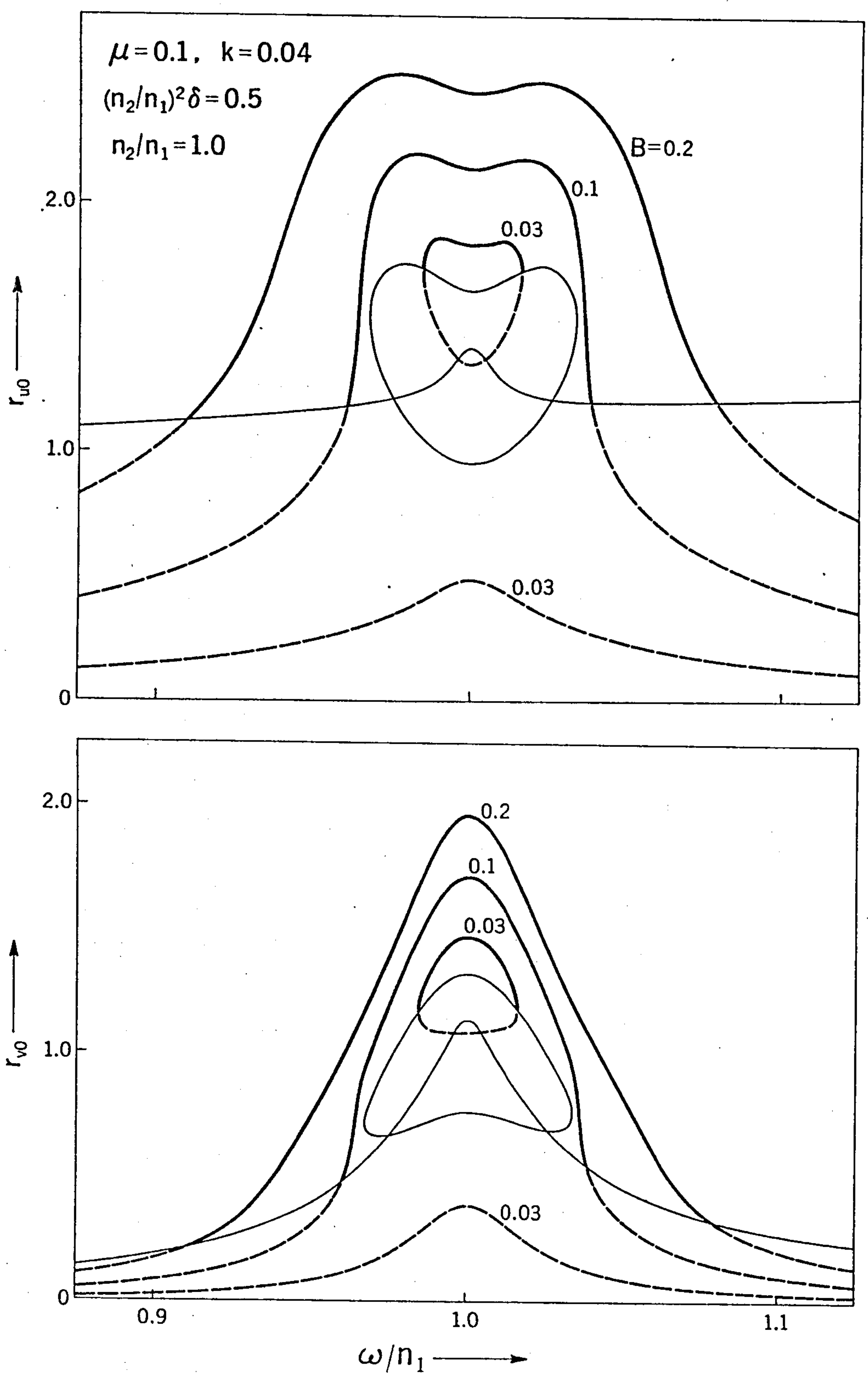

Fig. 6.1(a). Amplitude characteristic of the harmonic oscillation.

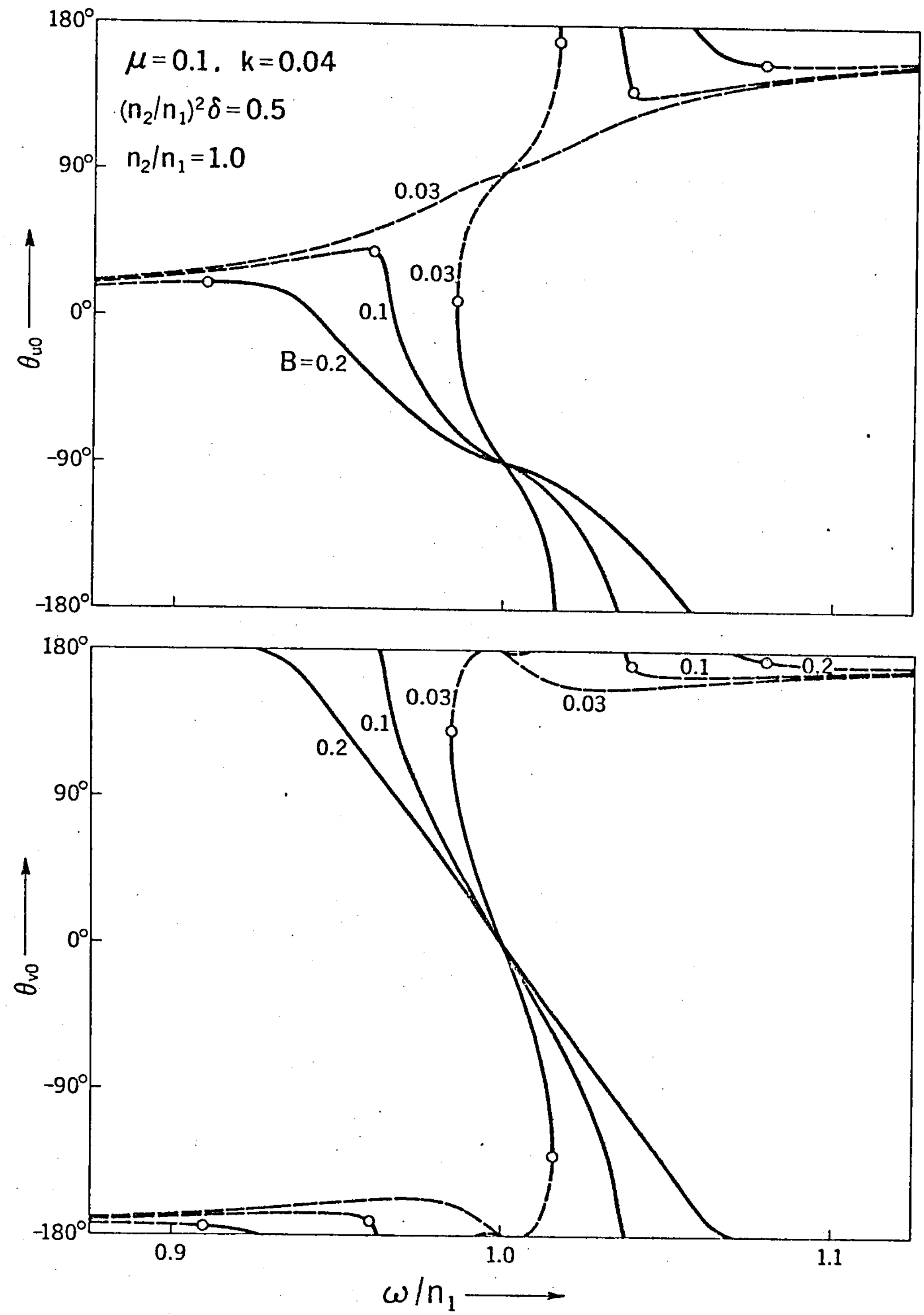

Fig. 6.1(b). Phase characteristic of the harmonic oscillation.

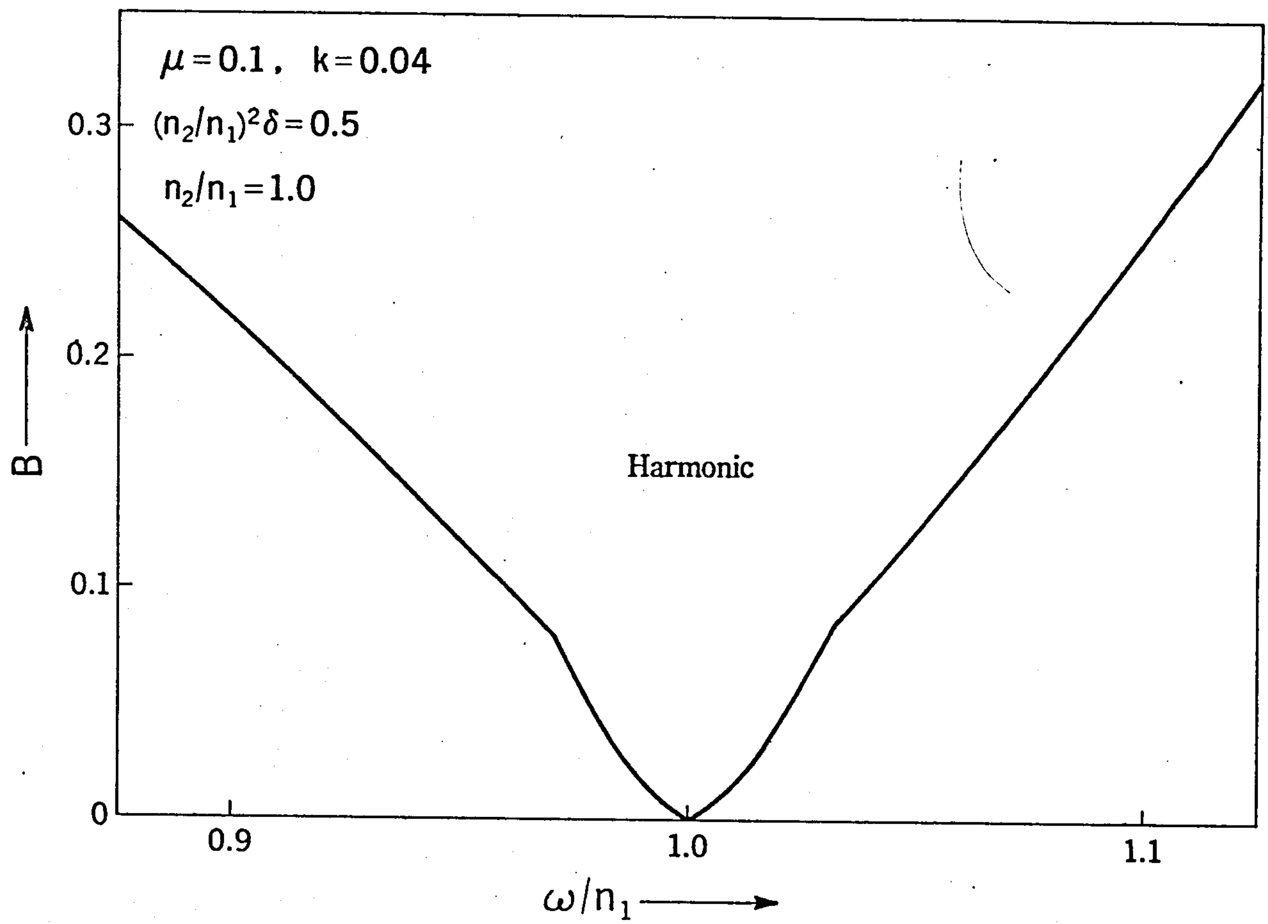

Fig. 6.1(c). Region of the harmonic entrainment.

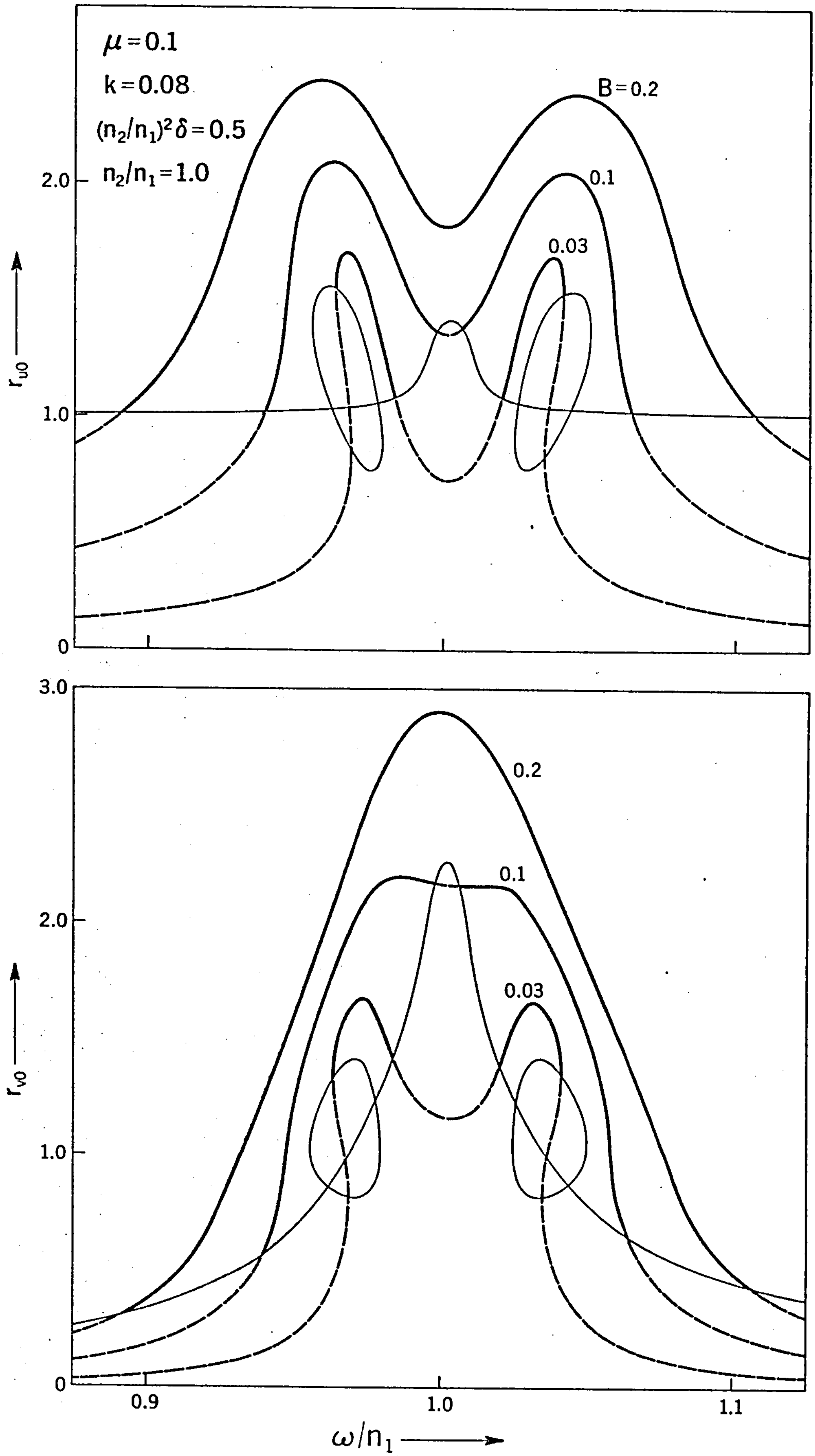

Fig. 6.2(a). Amplitude characteristic of the harmonic oscillation.

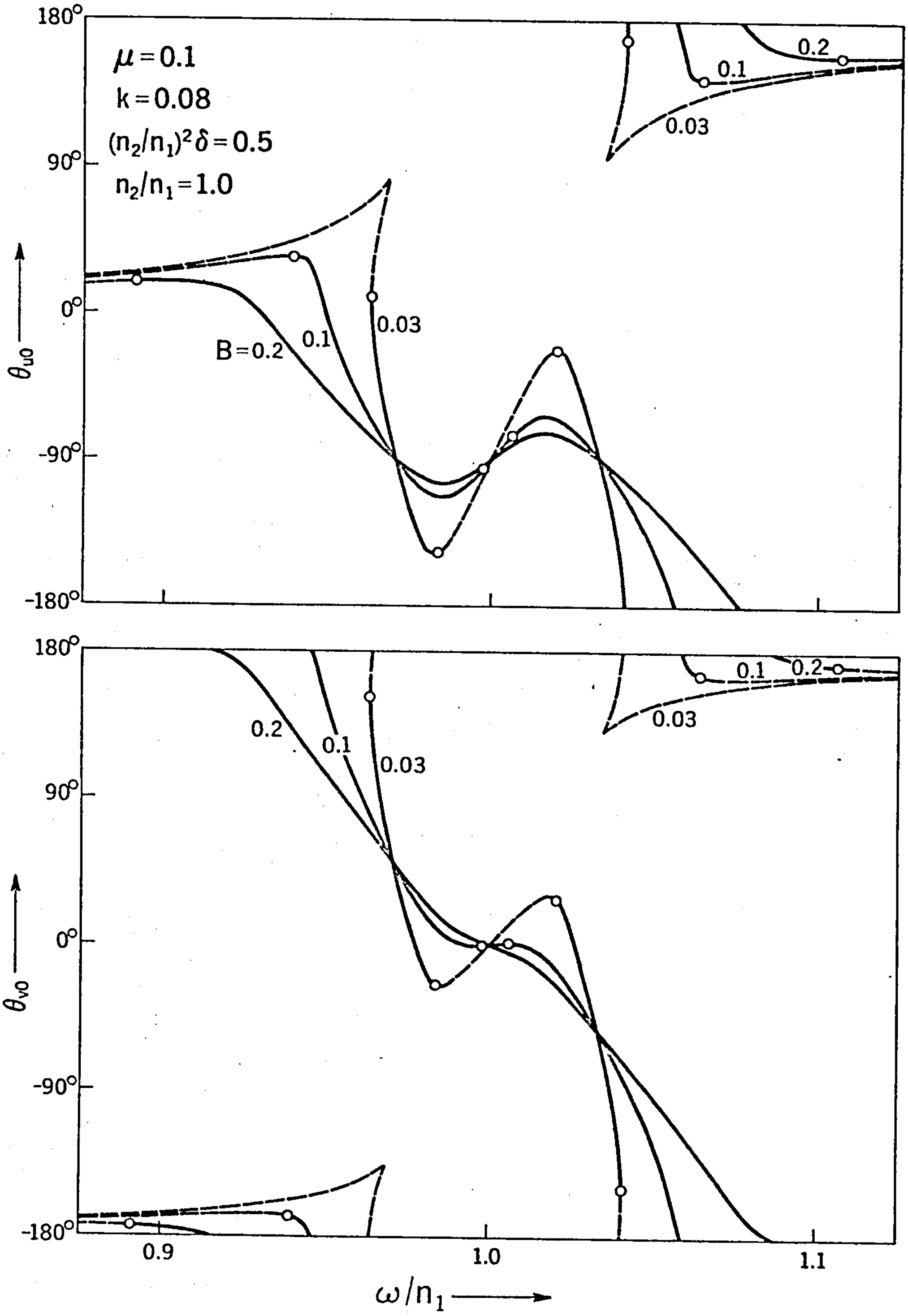

Fig. 6.2(b). Phase characteristic of the harmonic oscillation.

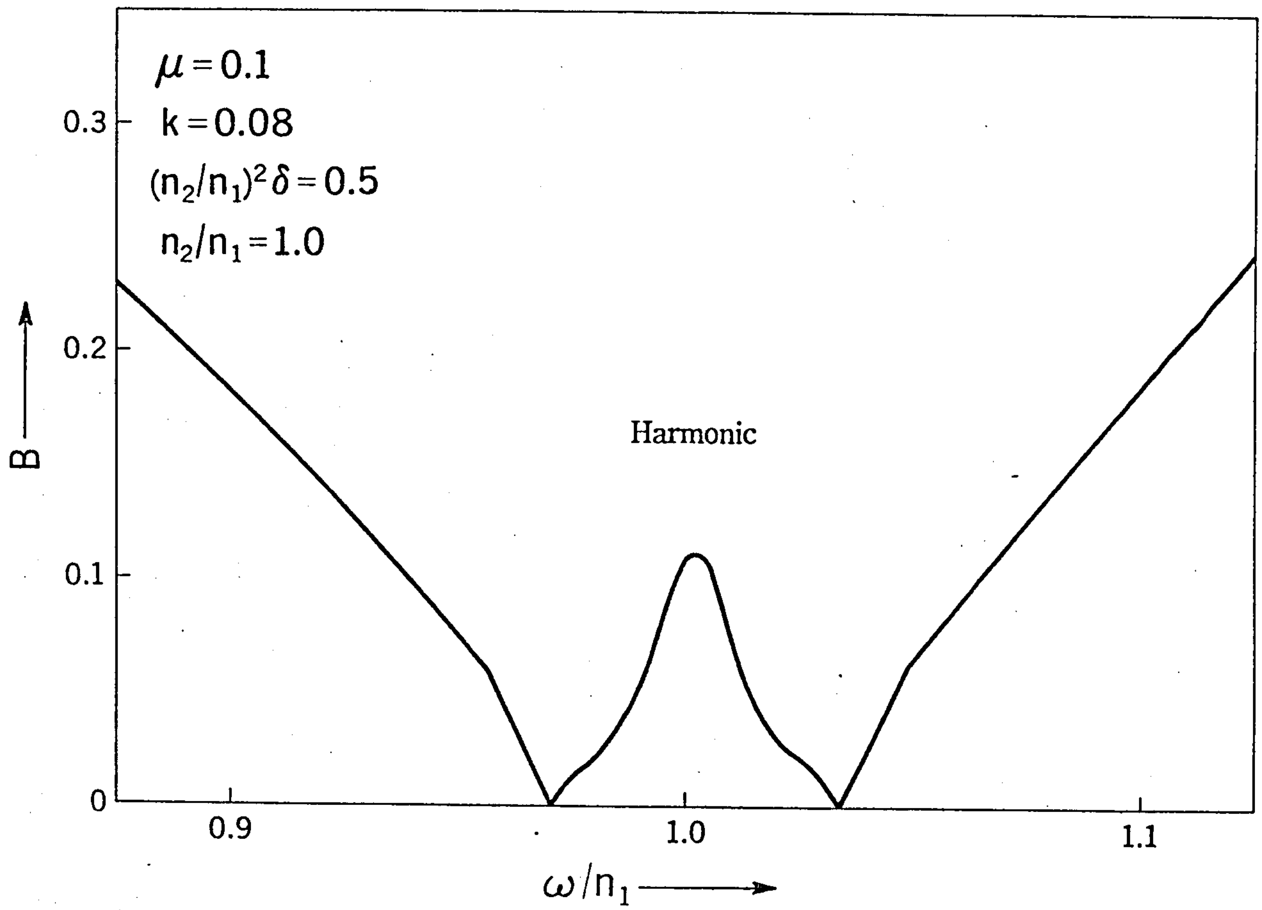

Fig. 6.2(c). Region of the harmonic entrainment.

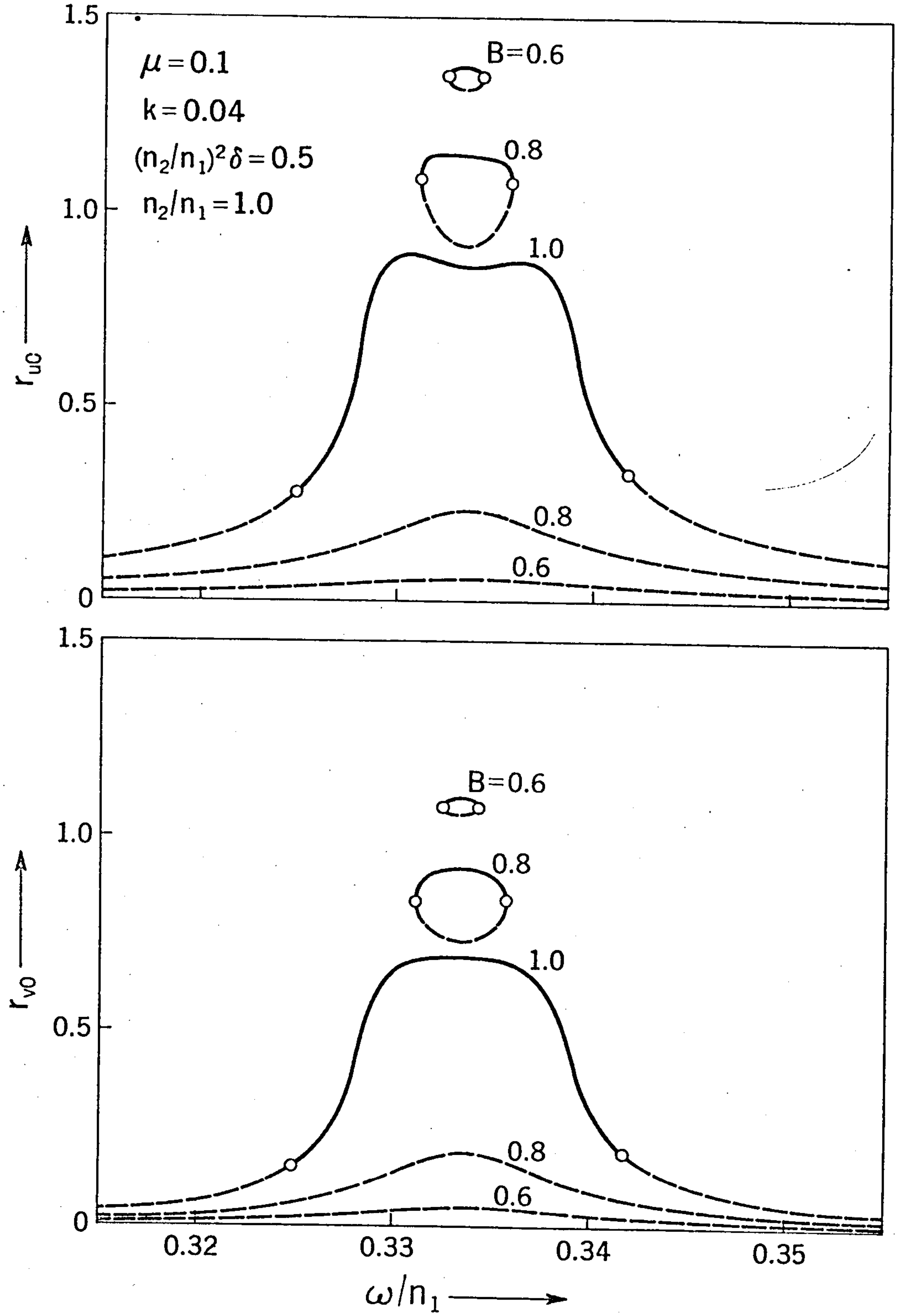

Fig. 6.3(a). Amplitude characteristic of the third-harmonic oscillation.

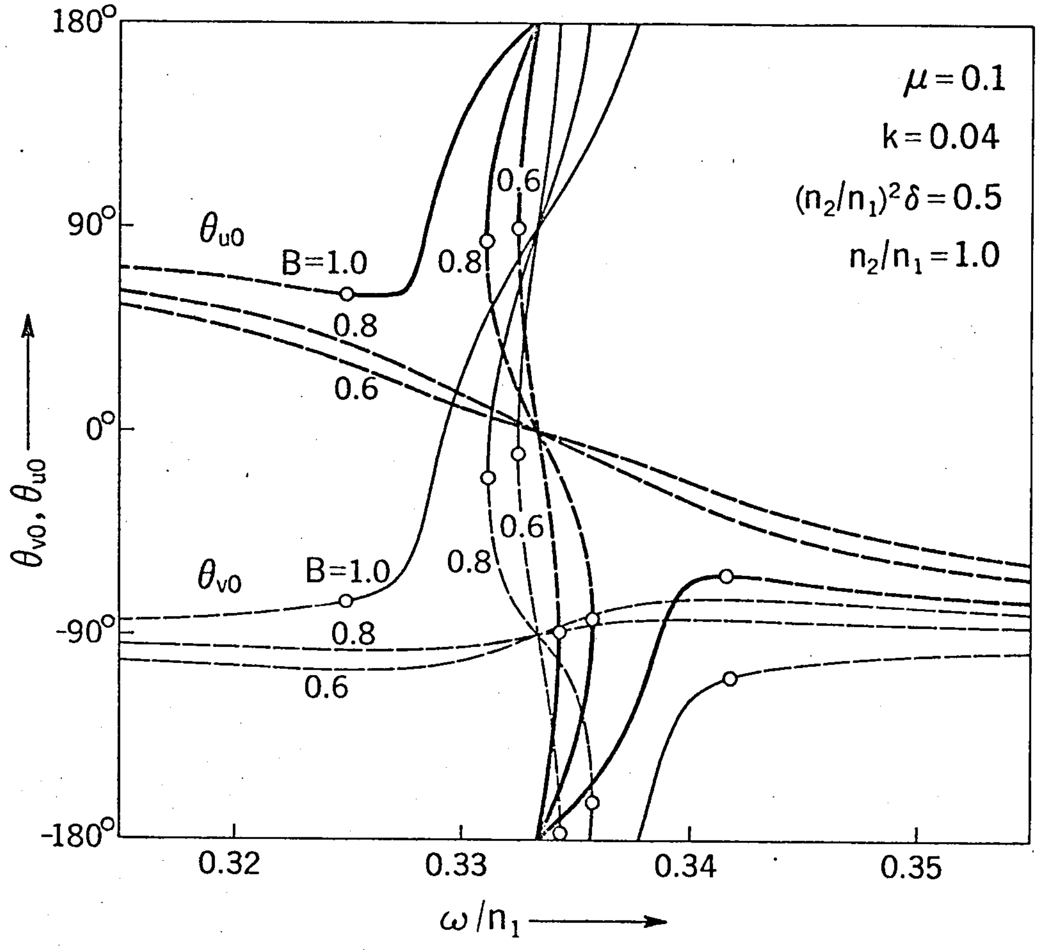

Fig. 6.3(b). Phase characteristic of the third-harmonic oscillation.

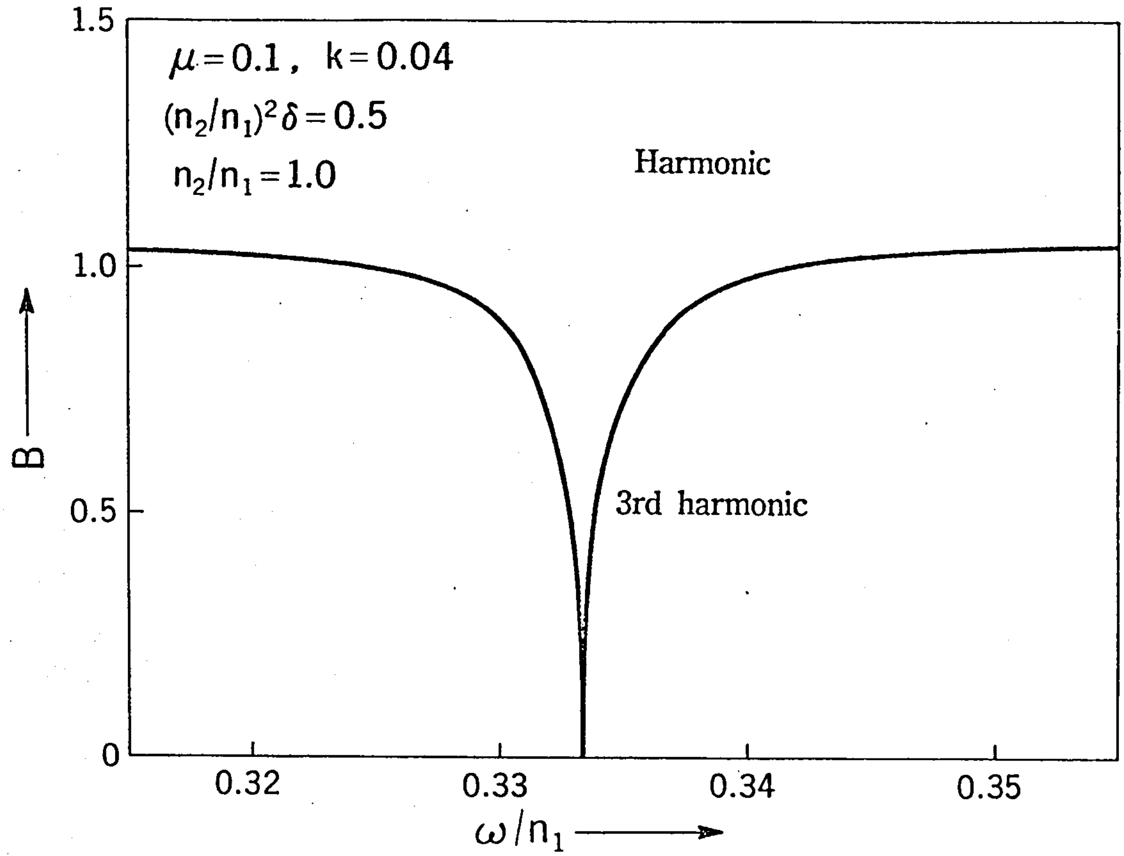

Fig. 6.3(c). Regions of the harmonic and third-harmonic entrainment.

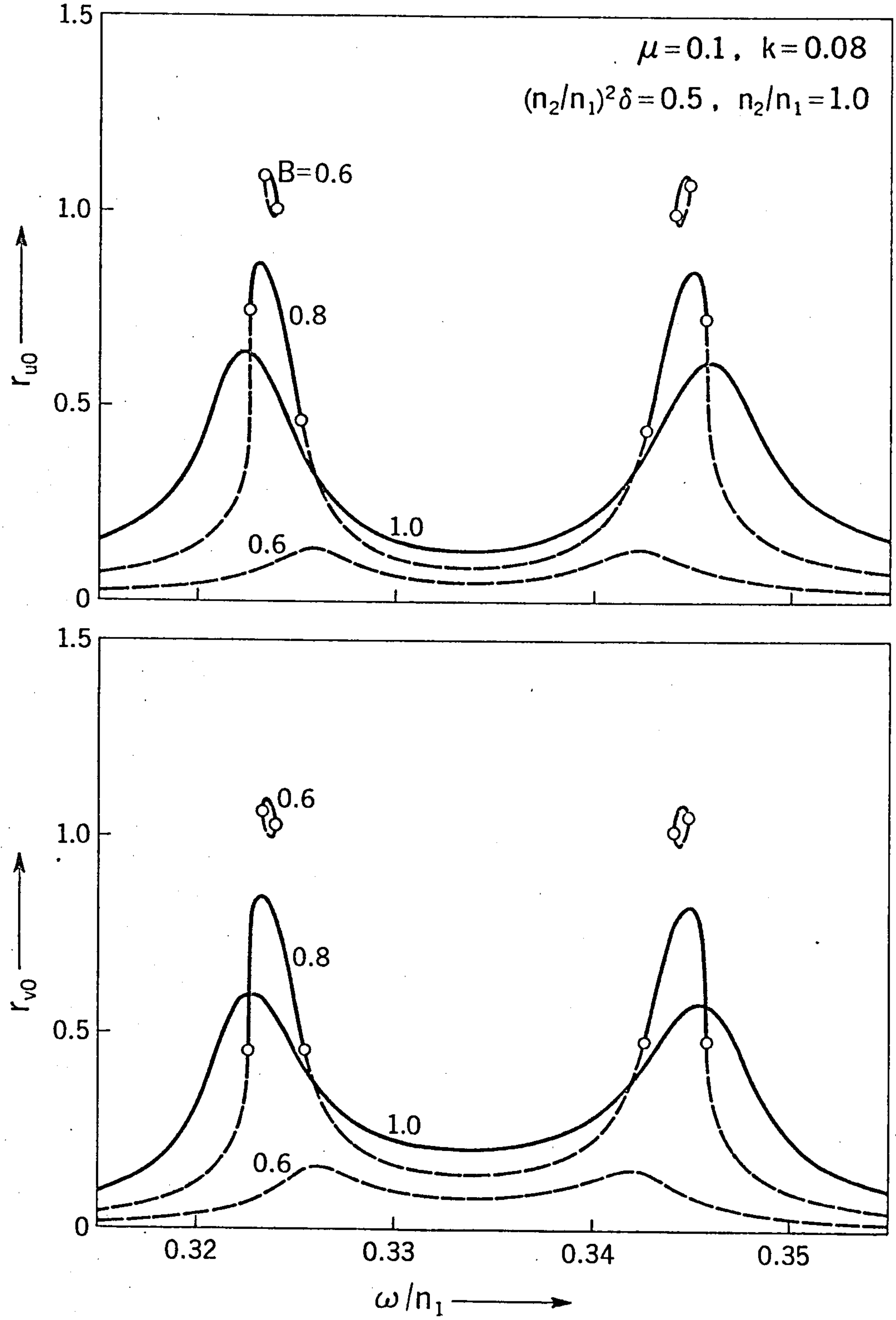

Fig. 6.4(a). Amplitude characteristic of the third-harmonic oscillation.

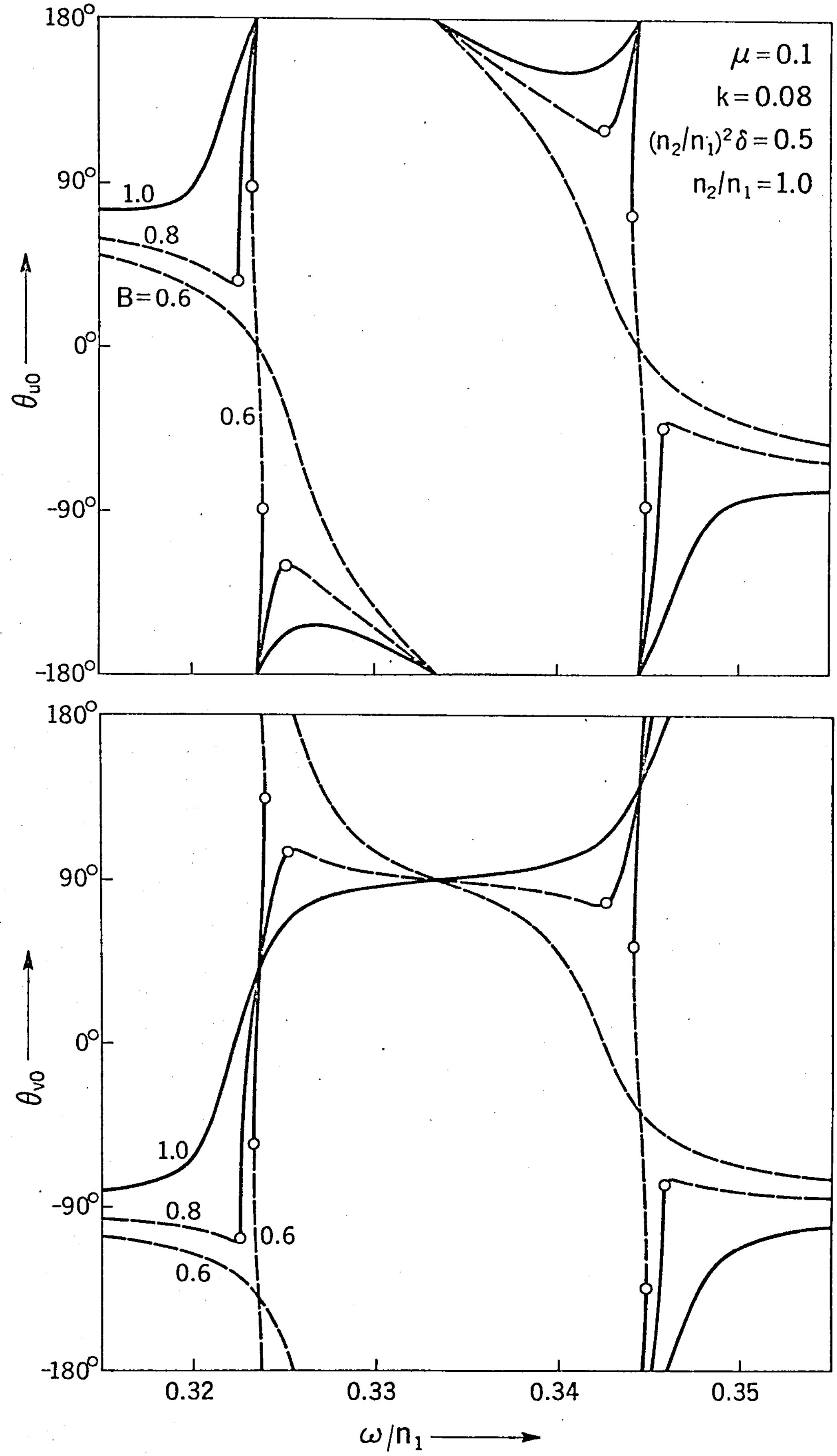

Fig. 6.4(b). Phase characteristic of the third-harmonic oscillation.

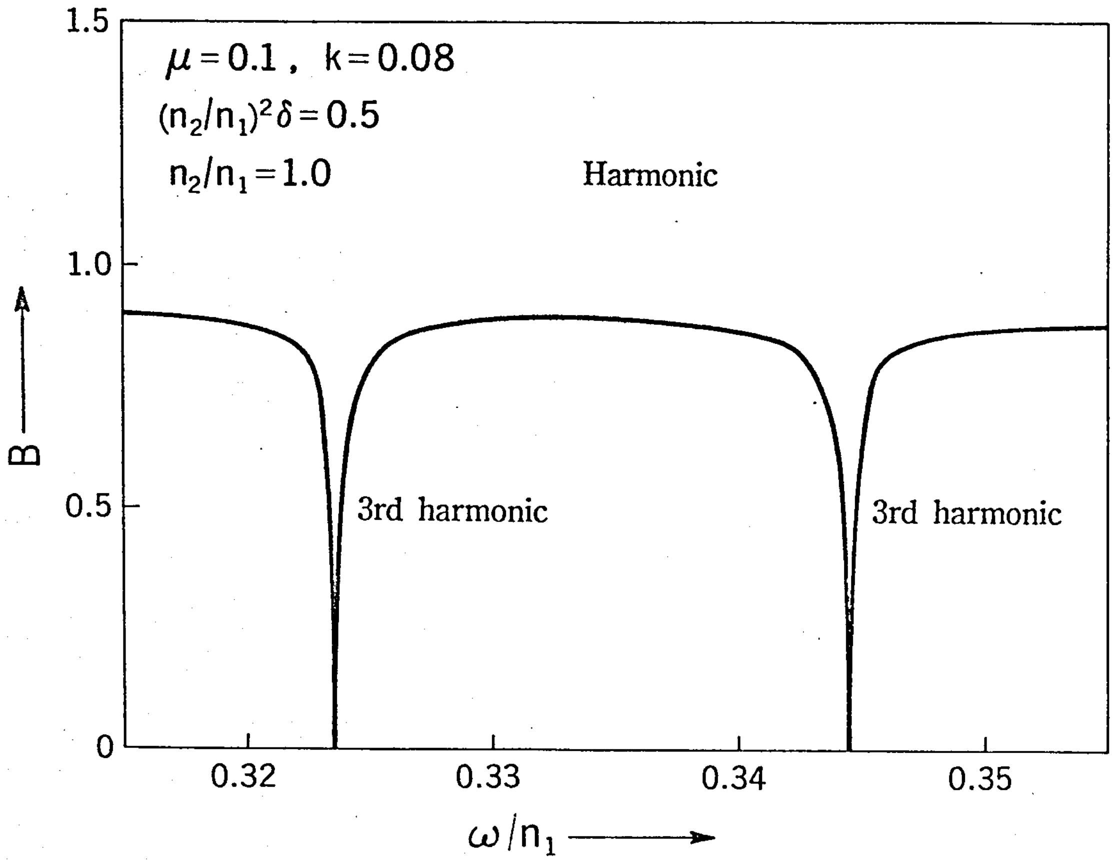

Fig. 6.4(c). Regions of the harmonic and third-harmonic entrainments.

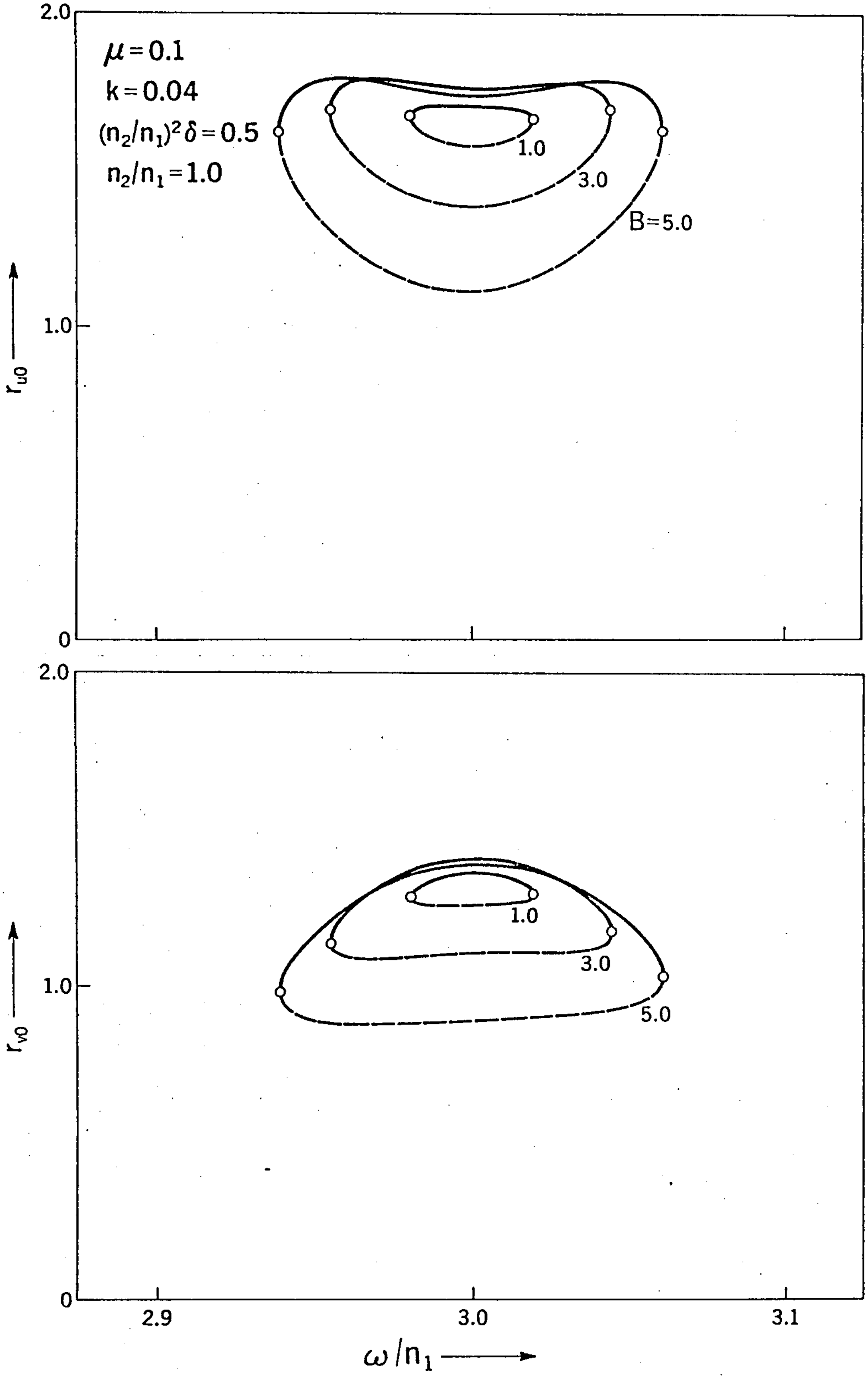

Fig. 6.5(a). Amplitude characteristic of the 1/3-harmonic oscillation.

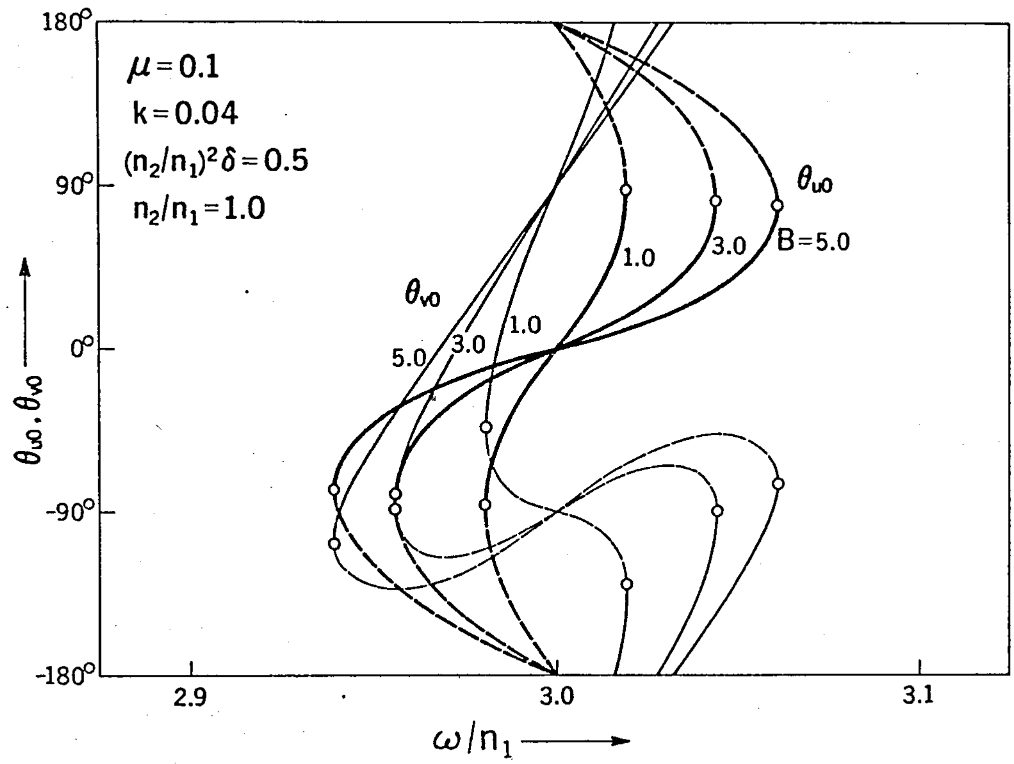

Fig. 6.5(b). Phase characteristic of the 1/3-harmonic oscillation.

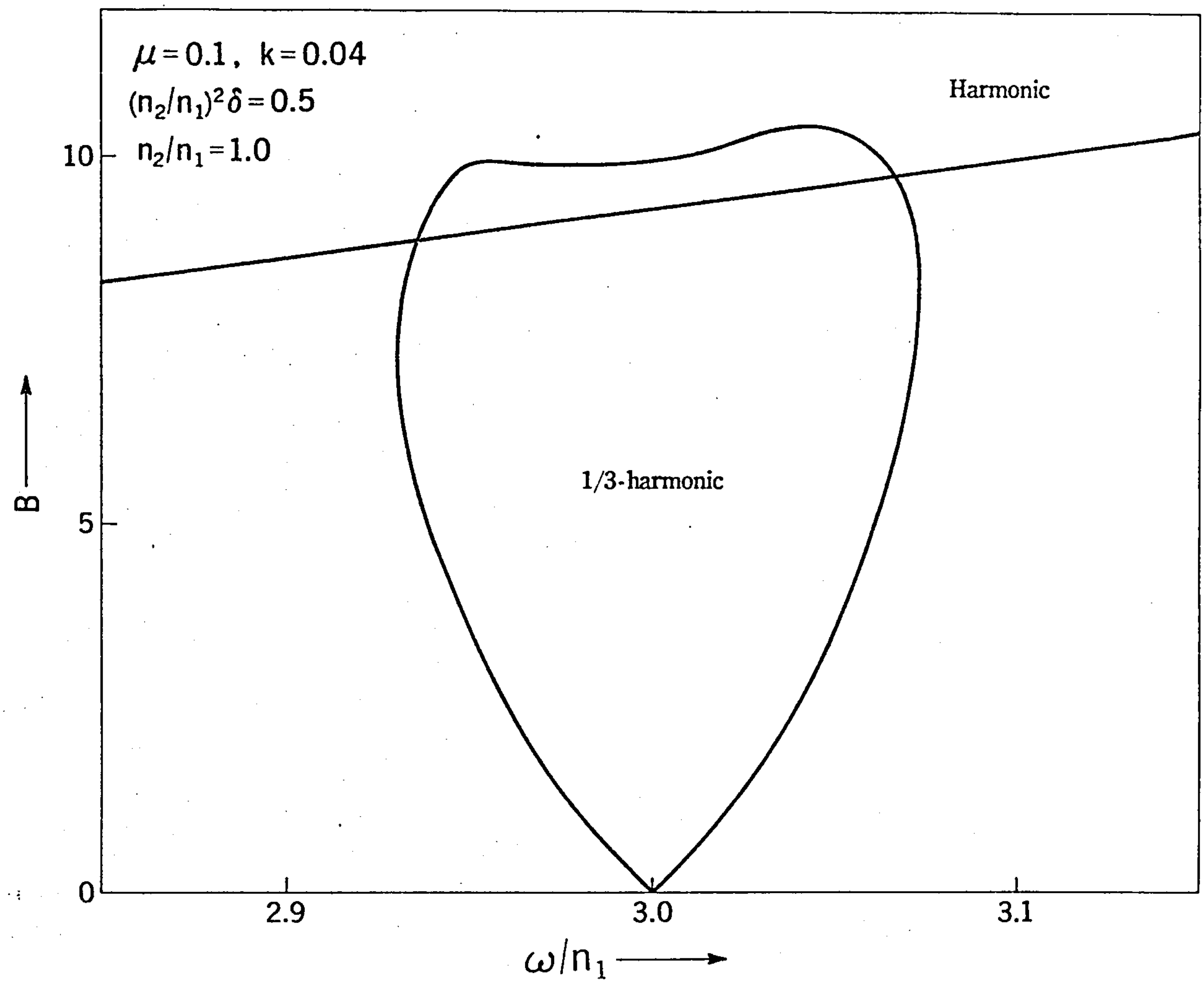

Fig. 6.5(c). Regions of the harmonic and 1/3-harmonic entrainments.

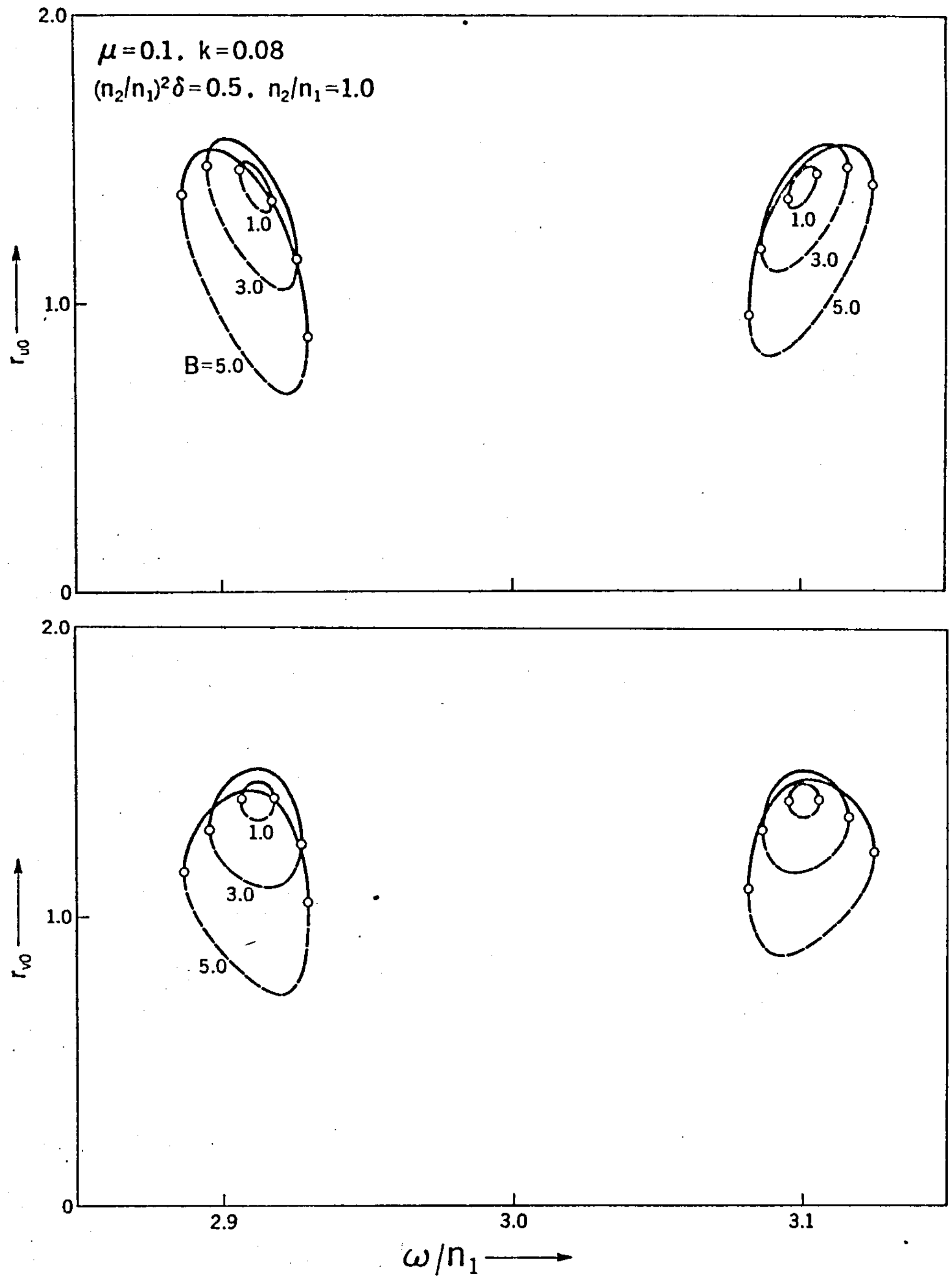

Fig. 6.6(a). Amplitude characteristic of the 1/3-harmonic oscillation.

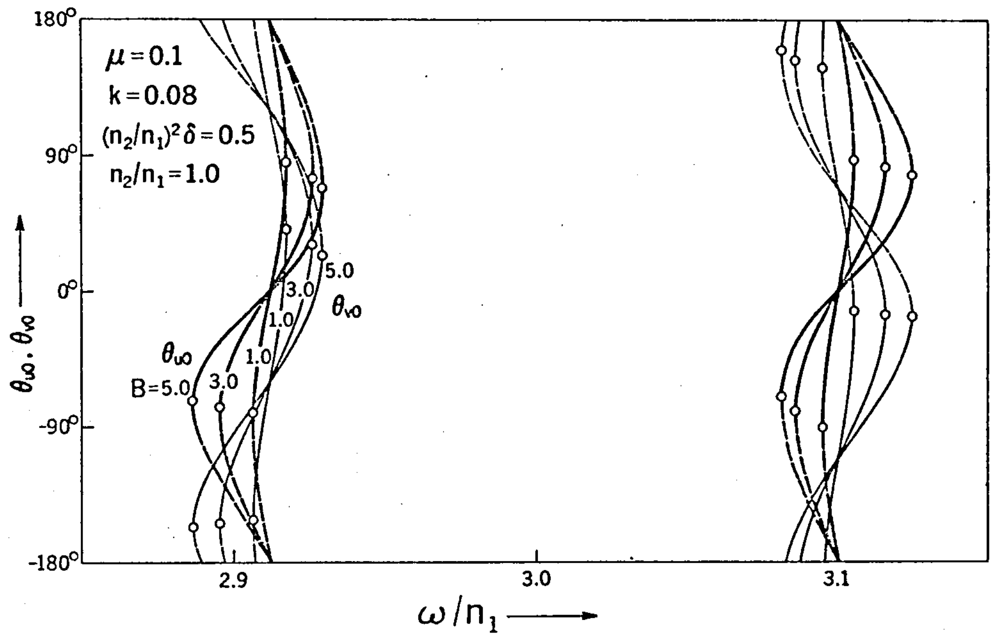

Fig. 6.6(b). Phase characteristic of the 1/3-harmonic oscillation.

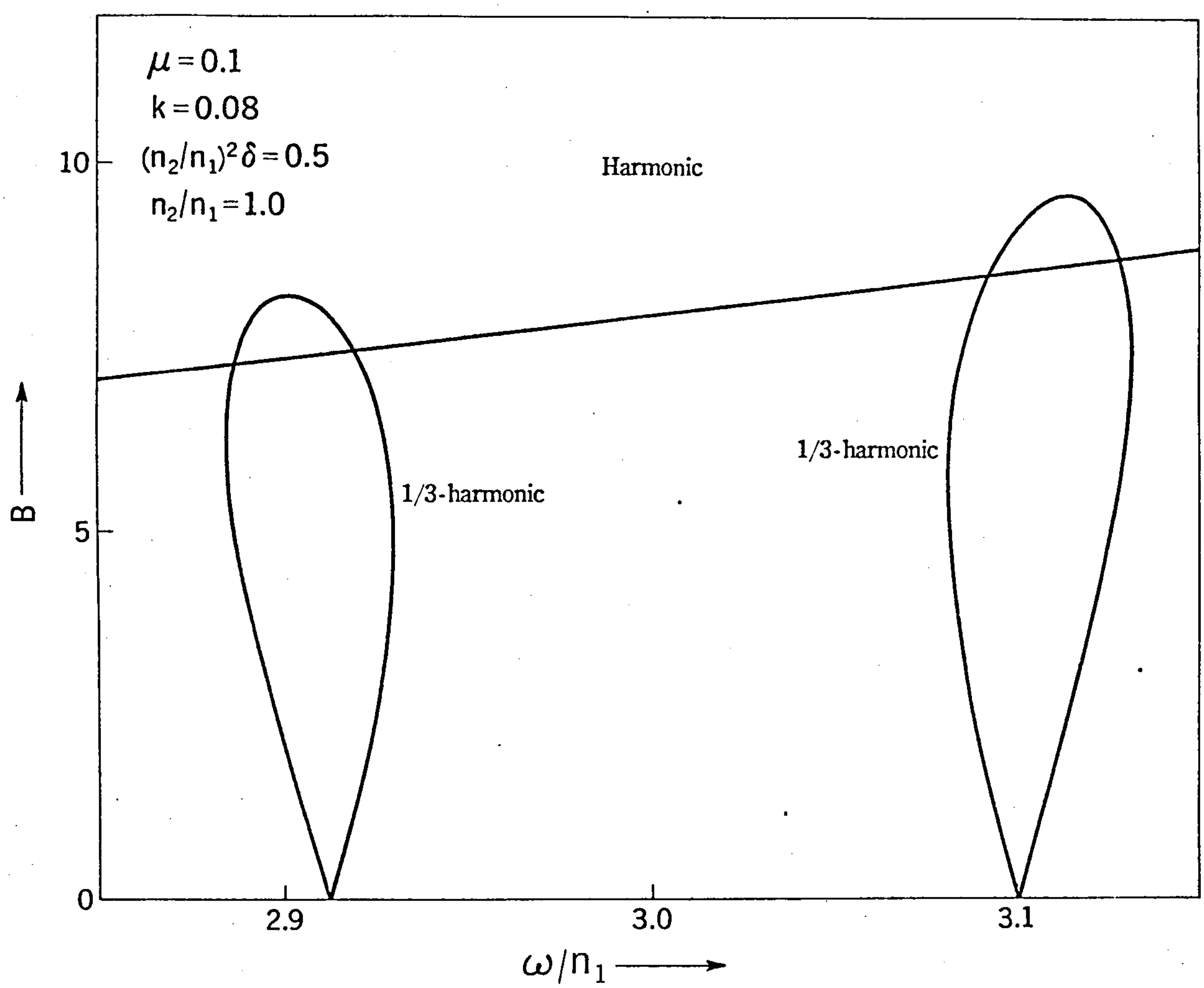

Fig. 6.6(c). Regions of the harmonic and 1/3-harmonic entrainments.

APPENDIX I

EXPANDED FORMS OF THE FUNDAMENTAL EQUATIONS

In the text we derived the autonomous equations of the first approximation from the fundamental equations by using the averaging method. In order to apply the averaging method, we introduced new variables r_i, θ_i $(i = 1, 2)$ and transformed the fundamental equations into those whose right-hand sides are multiplied by a small parameter μ, for instance, Eqs. (2.5) and (4.10). This appendix supplements the detailed forms of these equations.

We consider the following differential equations whose right-hand sides are expanded into trigonometric sums.

$$
\begin{aligned}
\dot{r}_1 &= -\frac{\mu}{\omega_{10}} f_n(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_{10} t + \theta_1) \\
&= \frac{\mu}{8} \frac{\omega_1^2}{n_1 \omega_{10}} \frac{k_2}{k_2 - k_1} \Big\{ \omega_{10}[\rho_1 - 2A_1^2 - r_1^2 - 2r_2^2] r_1 \\
&\quad + \frac{8 n_1 \omega_{10} \omega_{11}}{\omega_1^2} \frac{k_2 - k_1}{k_2} r_1 \sin 2\varphi_1 \\
&\quad - \omega_{10}[\rho_1 - 2A_1^2 - 2r_2^2] r_1 \cos 2\varphi_1 \\
&\quad - 2\omega_{10} r_1 r_2^2 \cos 2\varphi_2 \\
&\quad - 2\omega_{10} A_1^2 r_1 \cos 2\varphi_3 \\
&\quad - [\omega_{20}(\rho_0 - 2A_1^2 - r_2^2) + (2\omega_{10} - \omega_{20}) r_1^2] r_2 \cos(\varphi_1 + \varphi_2) \\
&\quad + [\omega_{20}(\rho_0 - 2A_1^2 - r_2^2) - (2\omega_{10} + \omega_{20}) r_1^2] r_2 \cos(\varphi_1 - \varphi_2) \\
&\quad - [4\omega(A_1 + \frac{\delta}{k_2} A_2) - \omega(A_1^2 + r_1^2 + 2r_2^2) A_1 + 2\omega_{10} A_1 r_1^2] \cos(\varphi_1 + \varphi_3) \\
&\quad + [4\omega(A_1 + \frac{\delta}{k_2} A_2) - \omega(A_1^2 + r_1^2 + 2r_2^2) A_1 - 2\omega_{10} A_1 r_1^2] \cos(\varphi_1 - \varphi_3)
\end{aligned}
$$

$$- 4\omega_{10}A_1r_1r_2 \cos(\varphi_2 + \varphi_3)$$

$$- 4\omega_{10}A_1r_1r_2 \cos(\varphi_2 - \varphi_3)$$

$$+ \omega_{20}r_2^3 \cos(\varphi_1 + 3\varphi_2)$$

$$- \omega_{20}r_2^3 \cos(\varphi_1 - 3\varphi_2)$$

$$+ (2\omega_{10} + \omega_{20})r_1^2r_2 \cos(3\varphi_1 + \varphi_2)$$

$$+ (2\omega_{10} - \omega_{20})r_1^2r_2 \cos(3\varphi_1 - \varphi_2)$$

$$+ \omega A_1^3 \cos(\varphi_1 + 3\varphi_3)$$

$$- \omega A_1^3 \cos(\varphi_1 - 3\varphi_3)$$

$$+ (2\omega_{10} + \omega)A_1r_1^2 \cos(3\varphi_1 + \varphi_3)$$

$$+ (2\omega_{10} - \omega)A_1r_1^2 \cos(3\varphi_1 - \varphi_3)$$

$$+ (\omega_{10} + 2\omega_{20})r_1r_2^2 \cos(2\varphi_1 + 2\varphi_2)$$

$$+ (\omega_{10} - 2\omega_{20})r_1r_2^2 \cos(2\varphi_1 - 2\varphi_2)$$

$$+ (\omega_{10} + 2\omega)A_1^2r_1 \cos(2\varphi_1 + 2\varphi_3)$$

$$+ (\omega_{10} - 2\omega)A_1^2r_1 \cos(2\varphi_1 - 2\varphi_3)$$

$$+ (\omega_{20} + 2\omega)A_1^2r_2 \cos(\varphi_1 + \varphi_2 + 2\varphi_3)$$

$$+ (\omega_{20} - 2\omega)A_1^2r_2 \cos(\varphi_1 + \varphi_2 - 2\varphi_3)$$

$$- (\omega_{20} - 2\omega)A_1^2r_2 \cos(\varphi_1 - \varphi_2 + 2\varphi_3)$$

$$- (\omega_{20} + 2\omega)A_1^2r_2 \cos(\varphi_1 - \varphi_2 - 2\varphi_3)$$

$$+ (2\omega_{20} + \omega)A_1r_2^2 \cos(\varphi_1 + 2\varphi_2 + \varphi_3)$$

$$- (2\omega_{20} - \omega)A_1r_2^2 \cos(\varphi_1 - 2\varphi_2 + \varphi_3)$$

$$+ (2\omega_{20} - \omega)A_1 r_2^2 \cos(\varphi_1 + 2\varphi_2 - \varphi_3)$$
$$- (2\omega_{20} + \omega)A_1 r_2^2 \cos(\varphi_1 - 2\varphi_2 - \varphi_3)$$
$$+ 2(\omega_{10} + \omega_{20} + \omega)A_1 r_1 r_2 \cos(2\varphi_1 + \varphi_2 + \varphi_3)$$
$$+ 2(\omega_{10} - \omega_{20} + \omega)A_1 r_1 r_2 \cos(2\varphi_1 - \varphi_2 + \varphi_3)$$
$$+ 2(\omega_{10} + \omega_{20} - \omega)A_1 r_1 r_2 \cos(2\varphi_1 + \varphi_2 - \varphi_3)$$
$$+ 2(\omega_{10} - \omega_{20} - \omega)A_1 r_1 r_2 \cos(2\varphi_1 - \varphi_2 - \varphi_3)$$
$$+ \omega_{10} r_1^3 \cos 4\varphi_1$$
$$- 4n_1 B_1 \sin(\varphi_1 + \varphi_3)$$
$$- 4n_1 B_1 \sin(\varphi_1 - \varphi_3)\Big\} \tag{I.1}$$

$$r_1 \dot{\theta}_1 = \frac{\mu}{\omega_{10}} g_n(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{10} t + \theta_1)$$
$$= \frac{\mu}{8} \frac{\omega_1^2}{n_1 \omega_{10}} \frac{k_2}{k_2 - k_1} \Big\{ \frac{8n_1 \omega_{10} \omega_{11}}{\omega_1^2} \frac{k_2 - k_1}{k_2} r_1$$
$$+ \omega_{10}[\rho_1 - 2A_1^2 - 2r_1^2 - 2r_2^2] \sin 2\varphi_1$$
$$+ \frac{8n_1 \omega_{10} \omega_{11}}{\omega_1^2} \frac{k_2 - k_1}{k_2} r_1 \cos 2\varphi_1$$
$$- 4\omega_{20} r_1 r_2^2 \sin 2\varphi_2$$
$$- 4\omega A_1^2 r_1 \sin 2\varphi_3$$
$$+ [\omega_{20}(\rho_0 - 2A_1^2 - r_2^2) - (2\omega_{10} + 3\omega_{20}) r_1^2] r_2 \sin(\varphi_1 + \varphi_2)$$
$$- [\omega_{20}(\rho_0 - 2A_1^2 - r_2^2) + (2\omega_{10} - 3\omega_{20}) r_1^2] r_2 \sin(\varphi_1 - \varphi_2)$$
$$+ [4\omega(A_1 + \frac{\delta}{k_2} A_2) - \omega(A_1^2 + 3r_1^2 + 2r_2^2)A_1 - 2\omega_{10} A_1 r_1^2] \sin(\varphi_1 + \varphi_3)$$
$$- [4\omega(A_1 + \frac{\delta}{k_2} A_2) - \omega(A_1^2 + 3r_1^2 + 2r_2^2)A_1 + 2\omega_{10} A_1 r_1^2] \sin(\varphi_1 - \varphi_3)$$

$$- 4(\omega_{20} + \omega)A_1 r_1 r_2 \sin(\varphi_2 + \varphi_3)$$

$$- 4(\omega_{20} - \omega)A_1 r_1 r_2 \sin(\varphi_2 - \varphi_3)$$

$$- \omega_{20} r_2^3 \sin(\varphi_1 + 3\varphi_2)$$

$$+ \omega_{20} r_2^3 \sin(\varphi_1 - 3\varphi_2)$$

$$- (2\omega_{10} + \omega_{20}) r_1^2 r_2 \sin(3\varphi_1 + \varphi_2)$$

$$- (2\omega_{10} - \omega_{20}) r_1^2 r_2 \sin(3\varphi_1 - \varphi_2)$$

$$- \omega A_1^3 \sin(\varphi_1 + 3\varphi_3)$$

$$+ \omega A_1^3 \sin(\varphi_1 - 3\varphi_3)$$

$$- (2\omega_{10} + \omega) A_1 r_1^2 \sin(3\varphi_1 + \varphi_3)$$

$$- (2\omega_{10} - \omega) A_1 r_1^2 \sin(3\varphi_1 - \varphi_3)$$

$$- (\omega_{10} + 2\omega_{20}) r_1 r_2^2 \sin(2\varphi_1 + 2\varphi_2)$$

$$- (\omega_{10} - 2\omega_{20}) r_1 r_2^2 \sin(2\varphi_1 - 2\varphi_2)$$

$$- (\omega_{10} + 2\omega) A_1^2 r_1 \sin(2\varphi_1 + 2\varphi_3)$$

$$- (\omega_{10} - 2\omega) A_1^2 r_1 \sin(2\varphi_1 - 2\varphi_3)$$

$$- (\omega_{20} + 2\omega) A_1^2 r_2 \sin(\varphi_1 + \varphi_2 + 2\varphi_3)$$

$$- (\omega_{20} - 2\omega) A_1^2 r_2 \sin(\varphi_1 + \varphi_2 - 2\varphi_3)$$

$$+ (\omega_{20} - 2\omega) A_1^2 r_2 \sin(\varphi_1 - \varphi_2 + 2\varphi_3)$$

$$+ (\omega_{20} + 2\omega) A_1^2 r_2 \sin(\varphi_1 - \varphi_2 - 2\varphi_3)$$

$$- (2\omega_{20} + \omega) A_1 r_2^2 \sin(\varphi_1 + 2\varphi_2 + \varphi_3)$$

$$- (2\omega_{20} - \omega) A_1 r_2^2 \sin(\varphi_1 + 2\varphi_2 - \varphi_3)$$

$$+ (2\omega_{20} - \omega)A_1 r_2^2 \sin(\varphi_1 - 2\varphi_2 + \varphi_3)$$
$$+ (2\omega_{20} + \omega)A_1 r_2^2 \sin(\varphi_1 - 2\varphi_2 - \varphi_3)$$
$$- 2(\omega_{10} + \omega_{20} + \omega)A_1 r_1 r_2 \sin(2\varphi_1 + \varphi_2 + \varphi_3)$$
$$- 2(\omega_{10} + \omega_{20} - \omega)A_1 r_1 r_2 \sin(2\varphi_1 + \varphi_2 - \varphi_3)$$
$$- 2(\omega_{10} - \omega_{20} + \omega)A_1 r_1 r_2 \sin(2\varphi_1 - \varphi_2 + \varphi_3)$$
$$- 2(\omega_{10} - \omega_{20} - \omega)A_1 r_1 r_2 \sin(2\varphi_1 - \varphi_2 - \varphi_3)$$
$$- \omega_{10} r_1^3 \sin 4\varphi_1$$
$$- 4n_1 B_1 \cos(\varphi_1 + \varphi_3)$$
$$- 4n_1 B_1 \cos(\varphi_1 - \varphi_3)\Big\} \tag{I.2}$$

$$\dot{r}_2 = -\frac{\mu}{\omega_{20}} g_n(r_1, r_2, \theta_1, \theta_2, t) \sin(\omega_{20}t + \theta_2) \tag{I.3}$$

$$r_2\dot{\theta}_2 = -\frac{\mu}{\omega_{20}} g_n(r_1, r_2, \theta_1, \theta_2, t) \cos(\omega_{20}t + \theta_2) \tag{I.4}$$

where

$$\begin{aligned} \varphi_1 &= \omega_{10}t + \theta_1, \quad \varphi_2 = \omega_{20}t + \theta_2 \\ \varphi_3 &= \omega t \\ \rho_0 &= 4(1 + \delta) \end{aligned} \tag{I.5}$$

The detailed forms of the differential equations (I.3) and (I.4) regarding $\dot{r}_2$ and $\dot{\theta}_2$ are obtained by interchanging r_1 and r_2, θ_1 and θ_2, ω_1 and ω_2, ω_{10} and ω_{20}, ω_{11} and ω_{21}, (therefore φ_1 and φ_2), k_1 and k_2 in Eqs. (I.1) and (I.2).

The four kinds of expanded forms of fundamental equations which have appeared in the text can be derived from Eqs. (I.1) through (I.4) according to the following conditions.

(A) The fundamental equations for the self-excited oscillation without internal resonance, i.e., the concrete forms of Eqs. (2.5) are given by putting

$$\begin{aligned} &f_n = f_1 && g_n = g_1 \\ &A_1 = 0 && B_1 = 0 \\ &\omega_{i1} = 0 && \omega_{i0} = \omega_i \quad (i = 1, 2) \end{aligned} \tag{I.6}$$

in Eqs. (I.1) through (I.4).

(B) The concrete forms of the fundamental equations for the self-excited oscillation with the internal resonance $3\omega_{10} = \omega_{20}$ are given by putting

$$\begin{aligned} &f_n = f_{10} && g_n = g_{10} \\ &A_1 = 0 && B_1 = 0 \end{aligned} \tag{I.7}$$

in Eqs. (I.1) through (I.4) [see Eqs. (3.7)]

(C) The fundamental equations for the forced oscillation in a case where the external resonance $\omega = \omega_{i0}$ $(i = 1, 2)$ occurs were given by Eqs. (4.10). The detailed forms of Eqs. (4.10) are given by putting

$$\begin{aligned} &f_n = f_{30} && g_n = g_{30} \\ &A_1 = 0 \end{aligned} \tag{I.8}$$

in Eqs. (I.1) through (I.4).

(D) The fundamental equations for the forced oscillation in a case where $\omega \neq \omega_{i0}$ $(i = 1, 2)$ are given by putting

$$\begin{aligned} &f_n = f_{20} && g_n = g_{20} \\ &B_1 = 0 \end{aligned} \tag{I.9}$$

in Eqs. (I.1) through (I.4) [see Eqs. (4.17)].

APPENDIX II

THE STABILITY CONDITIONS AND THE VERTICAL TANGENCY OF THE CHARACTERISTIC CURVE

In this appendix we describe the Routh-Hurwitz criterion for stability which is used in the text, and verify that the characteristic curve has a vertical tangent at the stability limit.

II.1 Routh-Hurwitz Criterion for Nonlinear Systems*

We consider a physical system described by a set of simultaneous differential equations

$$
\begin{aligned}
\frac{dx_1}{dt} &= X_1(x_1, x_2, \ldots, x_n ; p_1, p_2, \ldots, p_m) \\
\frac{dx_2}{dt} &= X_2(x_1, x_2, \ldots, x_n ; p_1, p_2, \ldots, p_m) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{dx_n}{dt} &= X_n(x_1, x_2, \ldots, x_n ; p_1, p_2, \ldots, p_m)
\end{aligned}
\tag{II.1}
$$

where t is the independent variable and functions $X_1, X_2, \ldots, X_n$ are generally nonlinear functions of the dependent variables $x_1, x_2, \ldots, x_n$ and the system parameters $p_1, p_2, \ldots, p_m$.†

The equilibrium states represented by the singular points of Eqs. (II.1) are obtained by equating

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \ldots = \frac{dx_n}{dt} = 0$$

---

* The expression and notation of the criterion in this appendix follows those of Hayashi [10].

i.e.,

$$
\begin{aligned}
&X_1(x_1, x_2, \ldots, x_n \,;\, p_1, p_2, \ldots, p_m) = 0 \\
&X_2(x_1, x_2, \ldots, x_n \,;\, p_1, p_2, \ldots, p_m) = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&X_n(x_1, x_2, \ldots, x_n \,;\, p_1, p_2, \ldots, p_m) = 0
\end{aligned}
\tag{II.2}
$$

Since Eqs. (II.2) are generally nonlinear algebraic equations, there exist a number of equilibrium states. Let us denote a set of equilibrium values of x's by $x_{10}, x_{20}, \ldots, x_{n0}$, and consider small variations ξ's denoted by

$$
\begin{aligned}
x_1 &= x_{10} + \xi_1 \\
x_2 &= x_{20} + \xi_2 \\
&\cdots\cdots\cdots \\
x_n &= x_{n0} + \xi_n .
\end{aligned}
\tag{II.3}
$$

Substituting Eqs. (II.3) into Eqs. (II.2) and discarding terms of order higher than the first in ξ's gives

$$
\begin{aligned}
\frac{d\xi_1}{dt} &= a_{11}\xi_1 + a_{12}\xi_2 + \ldots + a_{1n}\xi_n \\
\frac{d\xi_2}{dt} &= a_{21}\xi_1 + a_{22}\xi_2 + \ldots + a_{2n}\xi_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\xi_n}{dt} &= a_{n1}\xi_1 + a_{n2}\xi_2 + \ldots + a_{nn}\xi_n
\end{aligned}
\tag{II.4}
$$

---

† By making use of the averaging method, fundamental equations (1.7) and (1.9) are transformed to the equations of this form. In Eqs. (1.7) and (1.9) the dependent variables are represented by r_i's and θ_i's, while χ_i's, n_i's, μ, δ, B, and ω are the parameters of the system.

where a_{ij} stands for $\partial X_i/\partial x_j$ at the equilibrium state $x = x_{10}$, $x_2 = x_{20}, \ldots,$ $x_n = x_{n0}$. The characteristic equation may be written as

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{(II.5)}$$

When expanded, this nth-order determinant leads to an equation of the form

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0 \qquad \text{(II.6)}$$

It is to be noted that

$$a_n = \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{vmatrix}. \qquad \text{(II.7)}$$

If the real parts of all the roots of this characteristic equation are negative, the corresponding equilibrium state is stable. The signs of the real parts of the roots λ are known by making use of the Routh-Hurwitz criterion [15, 31]. In applying this criterion, we construct a set of n determinants set up from the coefficients of the nth-degree characteristic equation (II.6). These determinants are formed as follows:

$$\Delta_1 = |a_1|$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{II.8}$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \cdots \\ a_5 & a_4 & a_3 & a_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

The Routh-Hurwitz criterion states that the real parts of the roots λ are negative provided that all the coefficients a_0, $a_1, \ldots, a_n$ are positive and that all the determinants $\Delta_1, \Delta_2, \ldots, \Delta_n$ are also positive.

When n = 2, the conditions mentioned above are reduced to

$$a_0 > 0 \qquad a_1 > 0 \qquad a_2 > 0 \tag{II.9}$$

When n = 3, the conditions are

$$\begin{gathered} a_0 > 0 \qquad a_1 > 0 \quad (\text{or } a_2 > 0) \qquad a_3 > 0 \\ a_1 a_2 - a_0 a_3 > 0 \end{gathered} \tag{II.10}$$

When n = 4, the conditions are as follows:

$$\begin{gathered} a_0 > 0 \qquad a_1 > 0 \quad (\text{or } a_2 > 0) \\ a_3 > 0 \qquad a_4 > 0 \\ a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 > 0 \end{gathered} \tag{II.11}$$

II.2 The Vertical Tangency of the Characteristic Curve

The characteristic curves are obtained by solving Eqs. (II.2). We now discuss the vertical tangency of the characteristic curve. By differentiating Eqs. (II.2) with respect to p_i while holding p_j ($j = 1, \ldots, i - 1, i + 1, \ldots, m$) constant, we obtain

$$\frac{\partial X_1}{\partial x_1}\frac{dx_1}{dp_i} + \frac{\partial X_1}{\partial x_2}\frac{dx_2}{dp_i} + \dots + \frac{\partial X_1}{\partial x_n}\frac{dx_n}{dp_i} + \frac{\partial X_1}{\partial p_i} = 0$$

$$\frac{\partial X_2}{\partial x_1}\frac{dx_1}{dp_i} + \frac{\partial X_2}{\partial x_2}\frac{dx_2}{dp_i} + \dots + \frac{\partial X_2}{\partial x_n}\frac{dx_n}{dp_i} + \frac{\partial X_2}{\partial p_i} = 0 \qquad \text{(II.12)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial X_n}{\partial x_1}\frac{dx_1}{dp_i} + \frac{\partial X_n}{\partial x_2}\frac{dx_2}{dp_i} + \dots + \frac{\partial X_n}{\partial x_n}\frac{dx_n}{dp_i} + \frac{\partial X_n}{\partial p_i} = 0$$

Solving these simultaneous equations gives us

$$\frac{dx_j}{dp_i} = \frac{\Delta_j}{\Delta} \quad (j = 1, 2, \dots, n) \qquad \text{(II.13)}$$

where

$$\Delta = \begin{vmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \cdots & \frac{\partial X_1}{\partial x_n} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \cdots & \frac{\partial X_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_n}{\partial x_1} & \frac{\partial X_n}{\partial x_2} & \cdots & \frac{\partial X_n}{\partial x_n} \end{vmatrix} \qquad \text{(II.14)}$$

and Δ_j is the determinant formed by replacing the jth column of Δ by $-(\partial X_1/\partial p_i)$, $-(\partial X_2/\partial p_i), \dots, -(\partial X_n/\partial p_i)$. The determinant Δ is identical with the coefficient a_n of characteristic equation (II.6) and $a_n > 0$ is one of the stability conditions. Hence the characteristic curve ($p_i x_j$ relation) has a vertical tangent ($dx_j/dp_i \longrightarrow \infty$) at the stability limit $\Delta = 0$.

APPENDIX III

CONDITIONS FOR SELF-EXCITATION IN A SYSTEM WITH TWO DEGREES OF FREEDOM

The oscillator (Fig. 1.1) with two degrees of freedom are described by Eqs. (1.9), i.e.,

$$\begin{aligned} \ddot{u} - \chi_1 \ddot{v} + n_1^2 u &= \mu n_1 (1 - u^2)\dot{u} \\ \ddot{v} - \chi_2 \ddot{u} + n_2^2 v &= -\mu \frac{n_2^2}{n_1} \delta \dot{v} \end{aligned} \tag{III.1}$$

As mentioned in Chapters 2 and 3 this system has an equilibrium state at $u = 0$, $v = 0$. If this equilibrium state is unstable, a self-excited oscillation occurs. In this appendix we shall discuss the stability of this equilibrium state of the fundamental equations (III.1).

We consider small variations from the origin, i.e.,

$$\begin{aligned} u &= \xi_1 \qquad & \dot{u} &= \xi_2 \\ v &= \eta_1 \qquad & \dot{v} &= \eta_2 \end{aligned} \tag{III.2}$$

Substituting Eqs. (III.2) into Eqs. (III.1) and disregarding terms of order higher than the first in ξ_i and η_i ($i = 1, 2$) gives the following variational equations.

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -Kn_1^2 & K\mu n_1 & -K\chi_1 n_2^2 & -K\chi_1 \mu \frac{n_2^2}{n_1}\delta \\ 0 & 0 & 0 & 1 \\ -K\chi_1 n_1^2 & K\chi_2 \mu n_1 & -Kn_2^2 & -K\mu \frac{n_2^2}{n_1}\delta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} \tag{III.3}$$

where

$$K = 1/(1 - \chi_1\chi_2)$$

The characteristic equation is written as

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -Kn_1^2 & K\mu n_1 - \lambda & -K\chi_1 n_2^2 & -K\chi_1\mu \frac{n_2^2}{n_1}\delta \\ 0 & 0 & -\lambda & 1 \\ -K\chi_2 n_1^2 & K\chi_2\mu n_1 & -Kn_2^2 & -K\mu\frac{n_2^2}{n_1}\delta - \lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 1 \\ -Kn_1^2 & K\mu n_1 - \lambda \end{vmatrix} \cdot \begin{vmatrix} -\lambda & 1 \\ -Kn_2^2 & -K\mu\frac{n_2^2}{n_1}\delta - \lambda \end{vmatrix}$$

$$- K^2\chi_1\chi_2 \begin{vmatrix} -\lambda & 1 \\ n_1^2 & -\mu n_1 \end{vmatrix} \cdot \begin{vmatrix} -\lambda & 1 \\ n_2^2 & \mu\frac{n_2^2}{n_1}\delta \end{vmatrix}$$

$$= 0 \tag{III.4}$$

When expanded, this determinant leads to an equation of the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{III.5}$$

where

$$\begin{aligned} a_1 &= -K\mu n_1[1 - \frac{n_2}{n_1}^2\delta] \\ a_2 &= K(n_1^2 + n_2^2 - \mu^2 n_2^2\delta) \\ a_3 &= -K\mu n_1 n_2^2(1 - \delta) \\ a_4 &= Kn_1^2 n_2^2 \end{aligned} \tag{III.6}$$

The stability conditions are given by

$$a_1 > 0 \qquad a_3 > 0 \qquad a_4 > 0$$
$$a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 > 0 \tag{III.7}$$

By virtue of Eqs. (III.6), it is easily seen that the third condition is always satisfied.* Therefore, the conditions (III.7) are written as

$$\mu[(n_2/n_1)^2\delta - 1] > 0 \qquad \mu(\delta - 1) > 0$$

and (III.8)

$$\chi_1\chi_2n_1^2n_2^2(1 - \delta)^2 - \delta\{(n_1^2 - n_2^2)^2 + \mu^2n_1^2n_2^2(1 - \delta)[1 - (n_2/n_1)^2\delta]\} > 0$$

It is noted that the left-hand side of the last condition of (III.8) is identical with that of Eqs. (3.31) which is satisfied when the amplitudes of self-excited oscillations become zero. When the conditions (III.8) are not satisfied, the state ($u = 0$, $v = 0$) of Eqs. (III.1) becomes unstable and a self-excited oscillation occurs.

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* Instead of $a_1 > 0$, we may employ $a_2 > 0$ as one of the conditions. When the parameter μ is small, this condition is also satisfied.

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